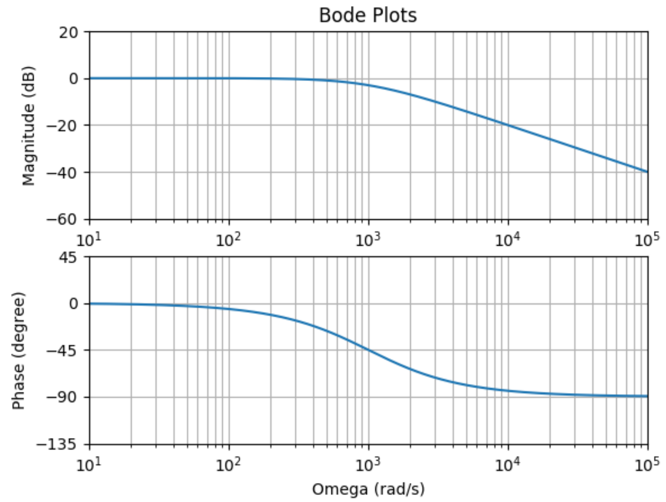
ELEC 2400 Electronic Circuits

Chapter 5: Frequency Response



Course Website: https://canvas.ust.hk

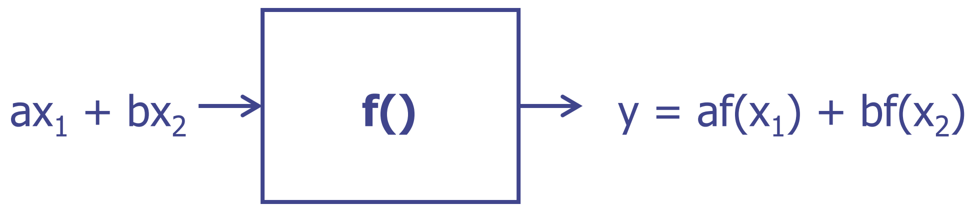
HKUST, 2021-22 Fall

Chapter 5: Frequency Response

Linearity (Review)

A circuit is linear iff it satisfies both the properties of homogeneity and superposition, that is

$$f(ax_1 + bx_2) = af(x_1) + bf(x_2)$$



Examples of linear circuit elements

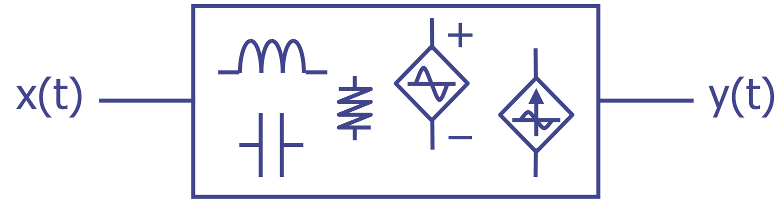
$$f(x) = ax \qquad \text{e.g.,} \qquad v = iR$$

$$f(x) = \frac{dx}{dt} \qquad v = L\frac{di}{dt}, \qquad i = C\frac{dv}{dt}$$

$$f(x) = \int x dt \qquad v = \frac{1}{C}\int i dt, \qquad i = \frac{1}{L}\int v dt$$

5.1 Frequency Response

Consider the linear network as shown below having an input voltage or current x(t) and an output voltage or current y(t).



Linear Network

The general equation for the output is

$$y = ax + b_1 \frac{d}{dt}x + b_2 \frac{d^2}{dt^2}x + \cdots + c_1 \int x dt + c_2 \iint x dt + \cdots$$

For a unit complex sinusoidal input of $x = e^{j\omega t}$, this simplifies to

$$y = ae^{j\omega t} + b_1 \frac{d}{dt}e^{j\omega t} + b_2 \frac{d^2}{dt^2}e^{j\omega t} + \cdots + c_1 \int e^{j\omega t} dt + c_2 \iint e^{j\omega t} dt + \cdots$$

$$= ae^{j\omega t} + b_1(j\omega)\, e^{j\omega t} + b_2(j\omega)^2 e^{j\omega t} + \cdots + \frac{c_1}{j\omega}e^{j\omega t} + \frac{c_2}{(j\omega)^2}e^{j\omega t} + \cdots$$

$$= \underbrace{\left[a + b_1(j\omega) + b_2(j\omega)^2 + \cdots + \frac{c_1}{j\omega} + \frac{c_2}{(j\omega)^2} + \cdots\right]}_{H(j\omega)} e^{j\omega t}$$

Frequency Response

The steady-state response to the unit complex sinusoidal input is therefore

$$y = H(j\omega)e^{j\omega t} = |H(j\omega)|e^{j\angle[H(j\omega)]+j\omega t}$$

Taking real parts, the steady-state AC input and output time signals are

$$\Re\mathrm{e}(x) = \cos(\omega t)$$

$$\Re\mathrm{e}(y) = |H(j\omega)| \cos\{\omega t + \angle[H(j\omega)]\}$$

Magnitude Phase Shift

Recall our phasor representation for x and y by omitting $e^{j\omega t}$

$$X = 1\angle 0°$$

$$Y = |H(j\omega)|\angle[H(j\omega)] = H(j\omega)$$

Hence

$$\frac{Y(j\omega)}{X(j\omega)} = H(j\omega)$$

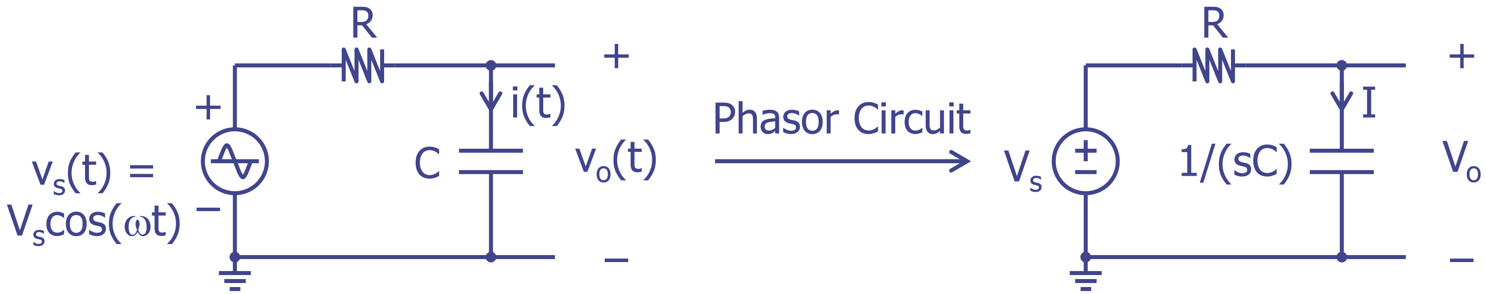
We write $X(j\omega)$ and $Y(j\omega)$ to remind us that frequency is the independent variable here. Further, let $s = j\omega$, we have

$$\frac{Y(s)}{X(s)} = H(s)$$

Transfer Function

Example 5-1

Example 5-1: Analyze the frequency response of the RC circuit below.



Soln.:

The resistor and capacitor impedances are R and 1/(sC), respectively. Together they form an AC voltage divider, with

$$V_o(s) = \left[\frac{\frac{1}{sC}}{R + \frac{1}{sC}} \right] V_s(s) = \left(\frac{1}{1 + sRC} \right) V_s(s)$$

$$\Rightarrow \qquad H(s) = \frac{V_o(s)}{V_s(s)} = \frac{1}{1 + sRC}$$

Example 5-1 (cont.)

Substituting $s = j\omega$

$$H(j\omega) = \frac{V_o(j\omega)}{V_s(j\omega)} = \frac{1}{1 + j\omega RC}$$

The magnitude response is

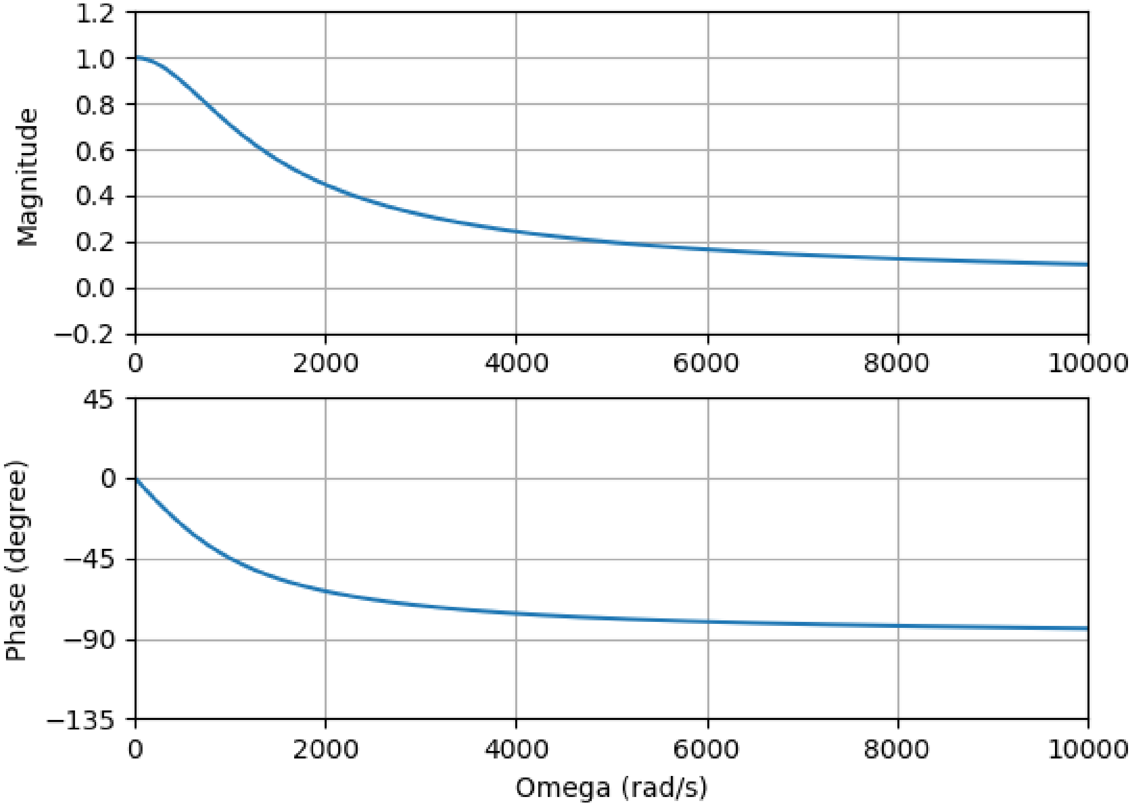
$$|H(j\omega)| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} = \begin{cases} 1 & \omega RC \ll 1 \\ \dfrac{1}{\omega RC} & \omega RC \gg 1 \end{cases}$$

The phase response is

$$\angle H(j\omega) = -\tan^{-1}(\omega RC) = \begin{cases} 0° & \omega RC \ll 1 \\ -90° & \omega RC \gg 1 \end{cases}$$

Example 5-1 (cont.)

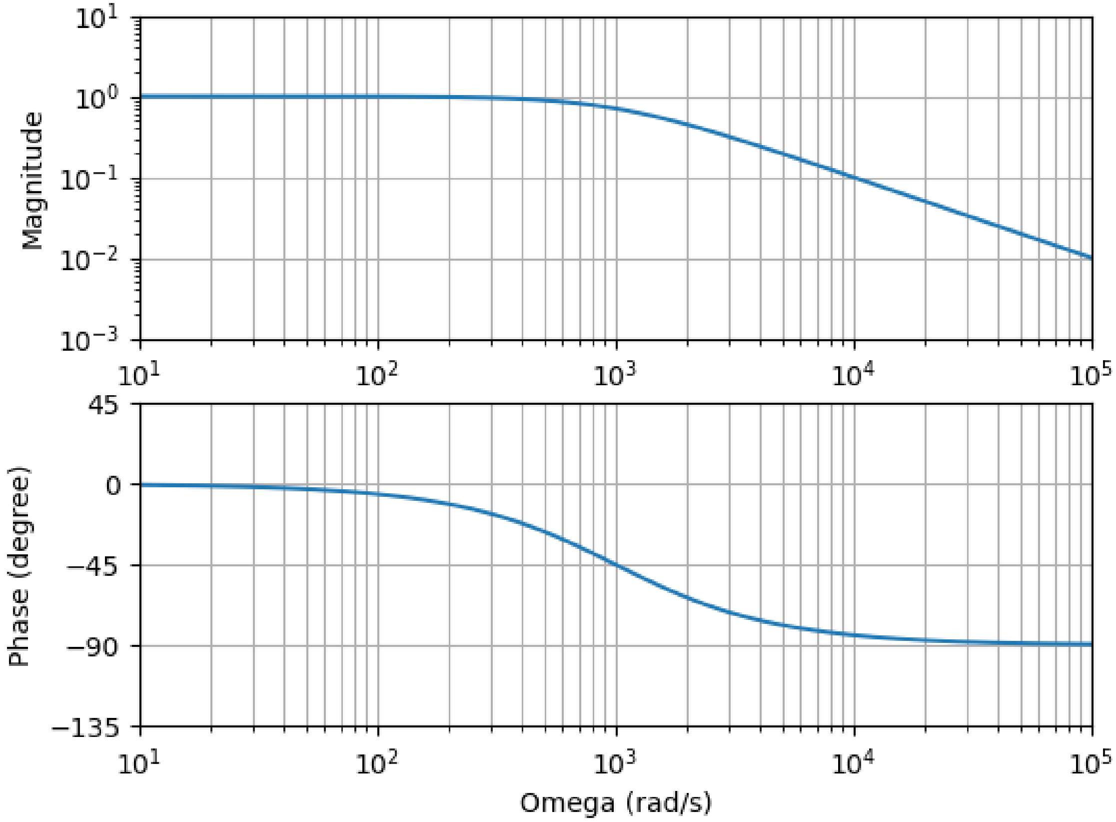
Let's examine the case when $RC = 1/1000$ (seconds). Thus, $\omega RC = 1$ when $\omega = 1000$ rad/s. In this case the magnitude and phase responses are



It turns out that linear plots are NOT the best way to elucidate the frequency response.

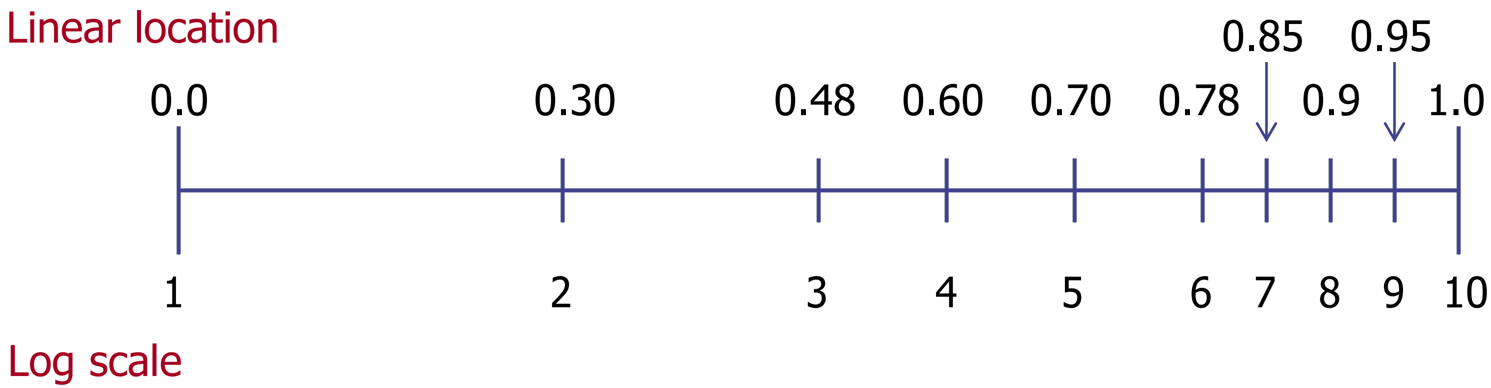
Example 5-1 (cont.)

Alternatively, we can use a log scale for both frequency and magnitude (via the dB concept to be explained later) but retain a linear scale for the phase angle.



Originally conceived by Hendrik Wade Bode, this is more revealing and is the standard way to plot frequency response.

Linear Scale vs Log Scale



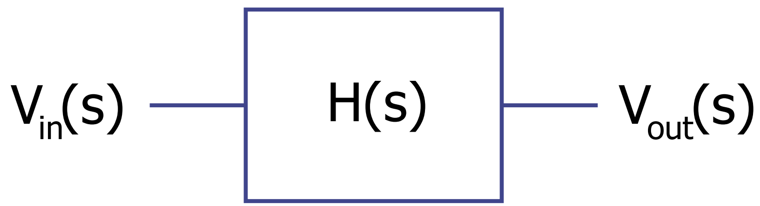
Log scale is a way of displaying numerical data over a very wide range of values in a compact way. It magnifies the lower end more than the higher end.

5.2 Transfer Function

An electrical circuit processes the input signal and give an output signal.

Transient analysis considers the time behavior of the input signal $v_{in}(t)$ and gives the output signal as $v_{out}(t)$.

In contrast, steady-state AC analysis considers an input signal at a particular frequency, for which we may write $v_{in}(t)$ as $V_{in}(s)$ ($s=j\omega$), and gives the output signal as $V_{out}(s)$. The ratio of $V_{out}(s)$ to $V_{in}(s)$ is called the transfer function $H(s)$ of the circuit.



$$H(s) = \frac{V_{out}(s)}{V_{in}(s)}$$

Formally, transfer function is defined through using the Laplace transform. Here we do not make this distinction, but simply change $j\omega$ to the Laplacian variable s ($s = j\omega$).

H(s) as Ratio of Polynomials in s

For a complicated linear circuit, the transfer function H(s) may be of very high order:

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{n_0 + n_1 s + \ldots + n_m s^m}{d_0 + d_1 s + \ldots + d_n s^n} = \frac{n(s)}{d(s)}$$

Do not confuse $V_{in}(s)$ with d(s), and $V_{out}(s)$ as n(s): strictly speaking, $V_{in}(s)$ and $V_{out}(s)$ are the Laplace transforms of the input and output signals, which themselves may be ratios of polynomials in s; whereas n(s) and d(s) are the numerator polynomial and the denominator polynomial of H(s).

The order of the system is the highest exponent in the transfer function. For example, if m = 2, n = 3, the system is of third order.

5.2.1 Poles and Zeros

Dealing with high order polynomials are very difficult, and it is much easier to handle first and second order systems one at a time. This is easily done as a k^{th}-order polynomial has k roots (fundamental theorem of algebra), and

$$H(s) = \frac{n(s)}{d(s)} = H_0 \frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

The roots of n(s) are $-z_1, -z_2, \ldots, -z_m$. Note that $n(s=-z_i) = 0$, and $H(s=-z_i) = 0$, so they are known as zeros of H(s).

The roots of d(s) are $-p_1, -p_2, \ldots, -p_n$. Note that $d(s=-p_j) = 0$, and $H(s=-p_j) = \infty$, so they are known as poles of H(s).

For a physical system, all coefficients n_i's and d_j's are real, and the roots have only two forms:

(1) some roots are real; and

(2) some roots are complex conjugate pairs.

Decomposing Magnitude and Phase of H(s)

In this course, we mainly focus on real roots, i.e., z_j and p_j are real. Let

$$(s+z_j) = |s+z_j| \angle\theta_i$$

$$(s+p_j) = |s+p_j| \angle\phi_i$$

Then

$$H(s) = |H(s)| \angle H(s) = H_0 \frac{|s+z_1|\angle\theta_1 \cdot |s+z_2|\angle\theta_2 \cdot \ldots \cdot |s+z_m|\angle\theta_m}{|s+p_1|\angle\phi_1 \cdot |s+p_2|\angle\phi_2 \cdot \ldots \cdot |s+p_n|\angle\phi_n}$$

Taking common logarithm, i.e., $\log_{10}$, on both sides, and separate the magnitude and the phase, thus turning multiplication and division into addition and subtraction:

$$\begin{aligned}\log|H(s)| &= \log H_0 + \log|s+z_1| + \log|s+z_2| + \ldots + \log|s+z_m| \\ &\quad - \log|s+p_1| - \log|s+p_2| - \ldots - \log|s+p_n|\end{aligned}$$

$$\angle H(s) = \angle\theta_1 + \angle\theta_2 + \ldots + \angle\theta_m - \angle\phi_1 - \angle\phi_2 - \ldots - \angle\phi_n$$

Hence, we analyze H(s) by analyzing |H(s)| and $\angle$H(s) separately.

Measure Power in dB

In comparing the ratio of two power levels, the common (base-10) logarithmic scale is usually used. The unit is the bel (B) after Alexander Graham Bell, the inventor of the telephone:

$$\frac{P_2}{P_1} = 10 \qquad \log\left(\frac{P_2}{P_1}\right) = \log(10) = 1B$$

$$\frac{P_2}{P_1} = 100 \qquad \log\left(\frac{P_2}{P_1}\right) = \log(100) = 2B$$

For historical reasons, engineers use decibel (dB) instead of bel, and 1 B = 10 dB. Hence,

$$\frac{P_2}{P_1} = 300 \qquad \log\left(\frac{P_2}{P_1}\right) = \log(300) = 24.77dB$$

In general,

$$\frac{P_2}{P_1} \quad \rightarrow \quad 10 \times \log\left(\frac{P_2}{P_1}\right) \quad dB$$

Measure Voltage and Current Ratios in 20×log dB

For voltages V_1 and V_2 driving a resistor R, the powers are

$$P_1 = \frac{V_1^{\,2}}{R} = I_1^{\,2}R \qquad\qquad P_2 = \frac{V_2^{\,2}}{R} = I_2^{\,2}R$$

Hence, the ratio of the two power levels is

$$\frac{P_2}{P_1} \rightarrow 10 \times \log\left(\frac{V_2^{\,2}/R}{V_1^{\,2}/R}\right) = 20 \times \log\frac{|V_2|}{|V_1|} \quad \text{dB}$$

$$\rightarrow 10 \times \log\left(\frac{I_2^{\,2}R}{I_1^{\,2}R}\right) = 20 \times \log\frac{|I_2|}{|I_1|} \quad \text{dB}$$

In circuit analysis, although the output voltage may be driving a different resistor from the input voltage, we still adopt the convention of

$$\frac{V_2}{V_1} \rightarrow 20 \times \log\frac{|V_2|}{|V_1|} \quad \text{dB}$$

5.2.2 Bode Plots

Bode plots turn a transfer function into its graphical representation. Bode plots consist of the magnitude plot and the phase plot. The actual curves can be approximated fairly accurately by using asymptotes.

For a single pole $\omega=p_1$ or zero at $\omega=z_1$:
For the magnitude plot, one asymptote is the $\log\omega$ axis, and the other one has a slope of ±20 dB/dec (dec=decade), that is, if the frequency changes by 10 times, the magnitude also changes by 10 times or ±20 dB. They meet at $\omega=p_1$ (z_1), forming a corner. Hence, p_1(z_1) is known as the corner frequency. Also, at $\omega=p_1$ (z_1), the magnitude is ±3 dB, and it is also known as the 3 dB frequency.

For the phase plot, the asymptotes are the $\log\omega$ axis that ends at $\omega=0.1p_1$(z_1), a horizontal line at ±90° that starts at $\omega=10p_1$ (z_1), and a line with a slope of ±45°/dec, starting from $0.1p_1$ (z_1) to $10p_1$ (z_1). The meeting points are at $\omega=0.1p_1$ (z_1) and $\omega=10p_1$ (z_1). The phase at $\omega=p_1$ (z_1) is ±45°.

Salient Features of Bode Plots for a Pole at $\omega=p_1$ or Zero at $\omega=z_1$

(1) The frequency axis is in log scale, i.e., log ω, but very often we just put down ω (and implicitly known to be in log scale).

(2) For the magnitude plot, the maximum error occurs at $\omega=p_1$ (z_1) when the asymptote reads 0 dB, but the actual curve should be ±3 dB.

(3) The phase at $\omega=p_1$ (z_1) is $\pm 45^o$.

(3) The maximum phase error using asymptotes is about 6^o.

(4) The Bode plots show the frequency response of the system.

Common Terminology in Bode Magnitude Plots

0 dB:

$|H(j\omega)| = 1$. Same power

3 dB:

$|H(j\omega)| = \sqrt{2}$, $20 \log |H(j\omega)| = 3.010$ dB. Double power

–3 dB:

$|H(j\omega)| = 1/\sqrt{2}$, $20 \log |H(j\omega)| = -3.010$ dB. Half power

20 dB/decade:

$|H(j\omega)|$ increases 10 times (power increases 100 times) when frequency goes up 10 times. Slope is +1 in the Bode magnitude log-log plot. Equivalent to 6 dB/octave.

–20 dB/decade:

$|H(j\omega)|$ decreases 10 times (power decreases 100 times) when frequency goes up 10 times. Slope is –1 in the Bode magnitude log-log plot. Equivalent to –6 dB/octave.

Magnitude and Phase of Real Zero

Consider $H(s) = 1 + s/z_1$, $s=j\omega$.

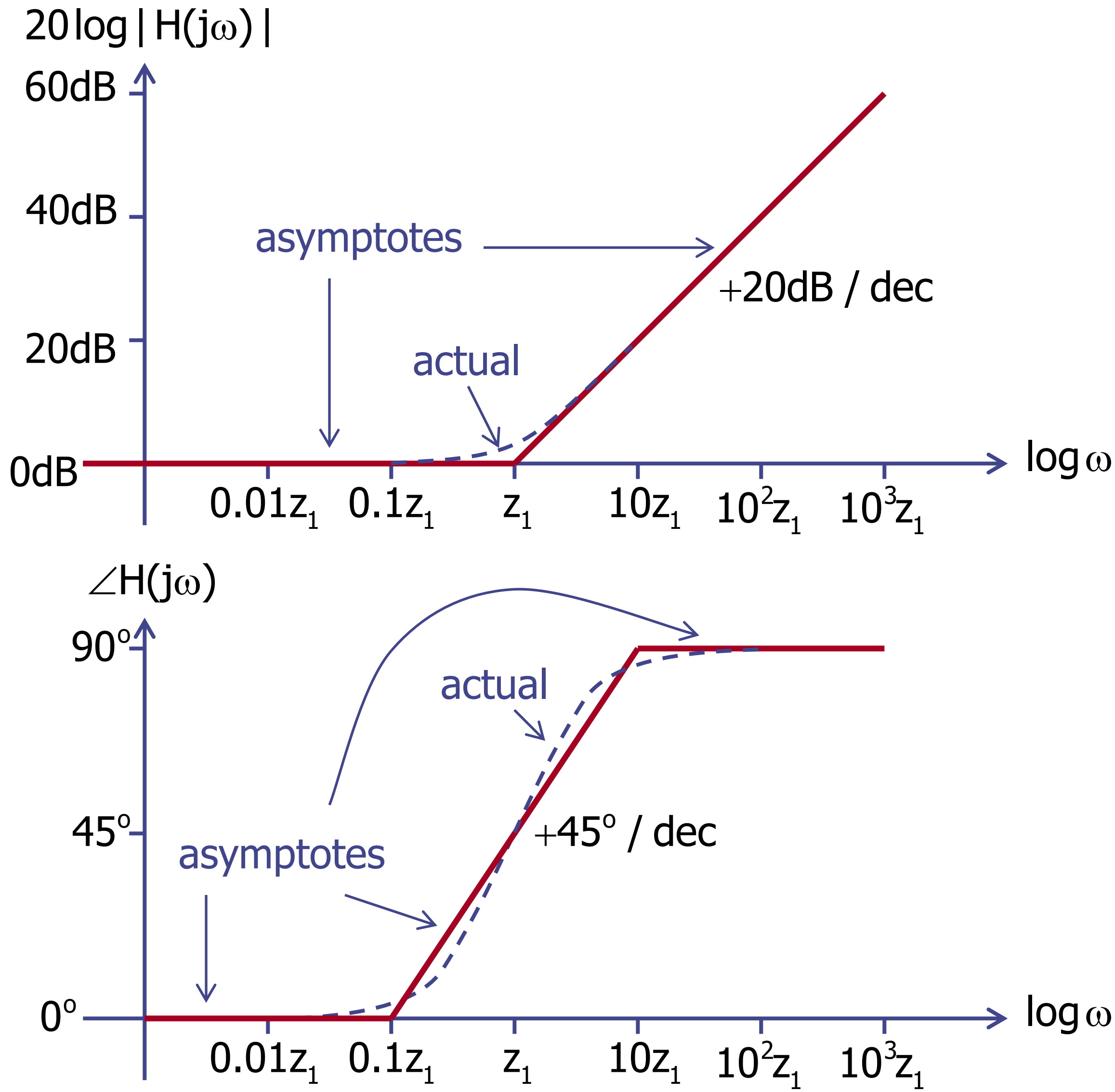
$H(j\omega)$	$\lvert H(j\omega)\rvert$	$20\log\lvert H(j\omega)\rvert$	$\angle H(j\omega)$
$H(j0)$	$=1$	0dB	0^o
$H(jz_1/100)$	$\sqrt{1+0.01^2} \approx 1$	0dB	0^o
$H(jz_1/10)$	$\sqrt{1+0.1^2} \approx 1$	0dB	5.7^o
$H(jz_1/3)$	$\sqrt{1+0.333^2} = 1.054 \approx 1$	0dB	18.4^o
$H(jz_1)$	$\sqrt{1+1} = \sqrt{2} \approx 1.414$	3dB	45^o
$H(j3z_1)$	$\sqrt{1+3^2} = \sqrt{10} \approx 3.162$	10dB	71.6^o
$H(j10z_1)$	$\sqrt{1+10^2} = \sqrt{101} \approx 10$	20dB	84.3^o
$H(j10^2 z_1)$	$\sqrt{1+100^2} \approx 100$	40dB	90^o
$H(j10^3 z_1)$	$\sqrt{1+1000^2} \approx 1000$	60dB	90^o

Magnitude and Phase Plots of Real Zero

$$H(s) = 1 + \frac{s}{z_1}$$

Magnitude plot:

Phase plot:



Magnitude and Phase Plots of Real Pole

$$H(s) = \frac{1}{1 + \frac{s}{p_1}}$$

Let

$$H'(s) = 1 + \frac{s}{p_1}$$

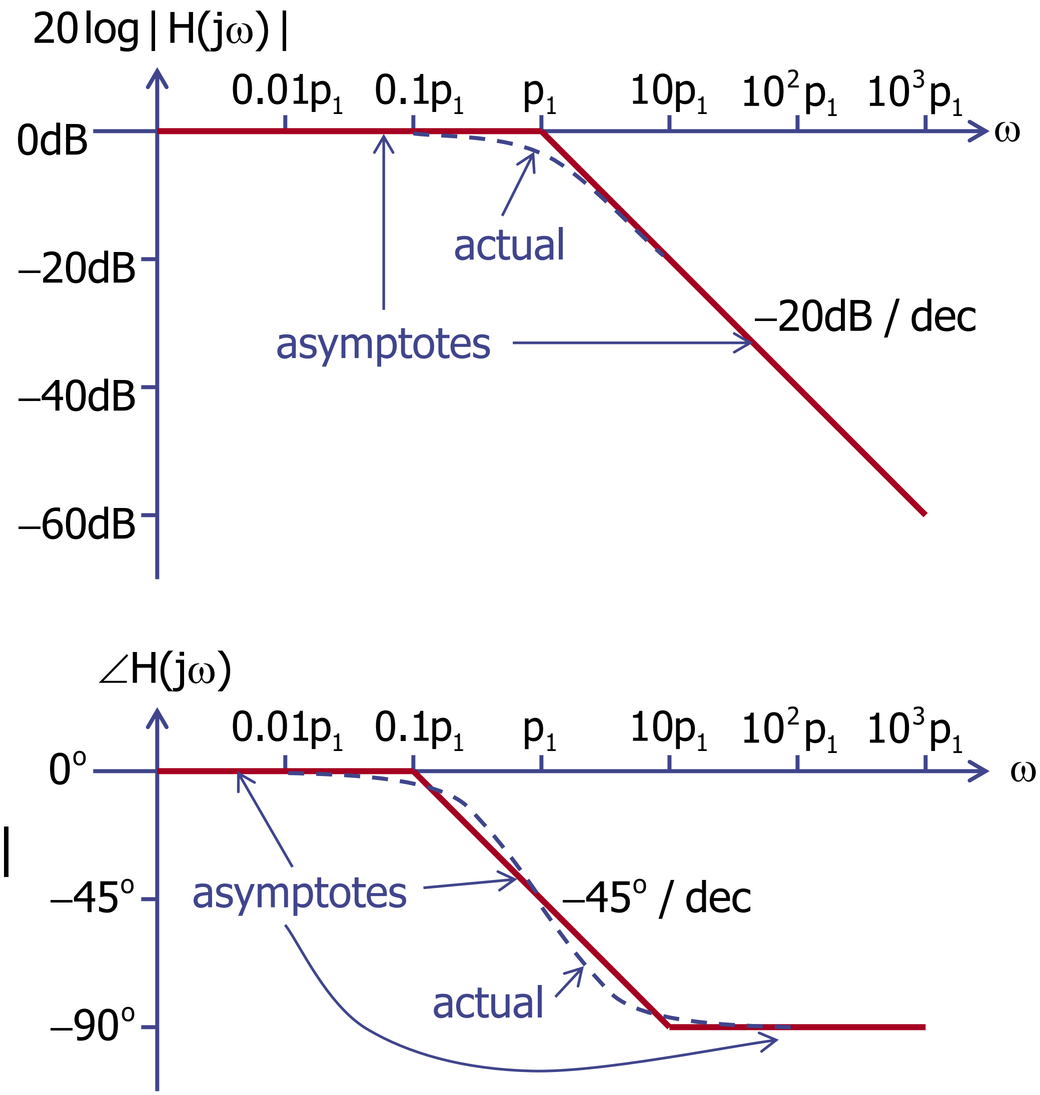
then

$$H(s) = \frac{1}{H'(s)}$$

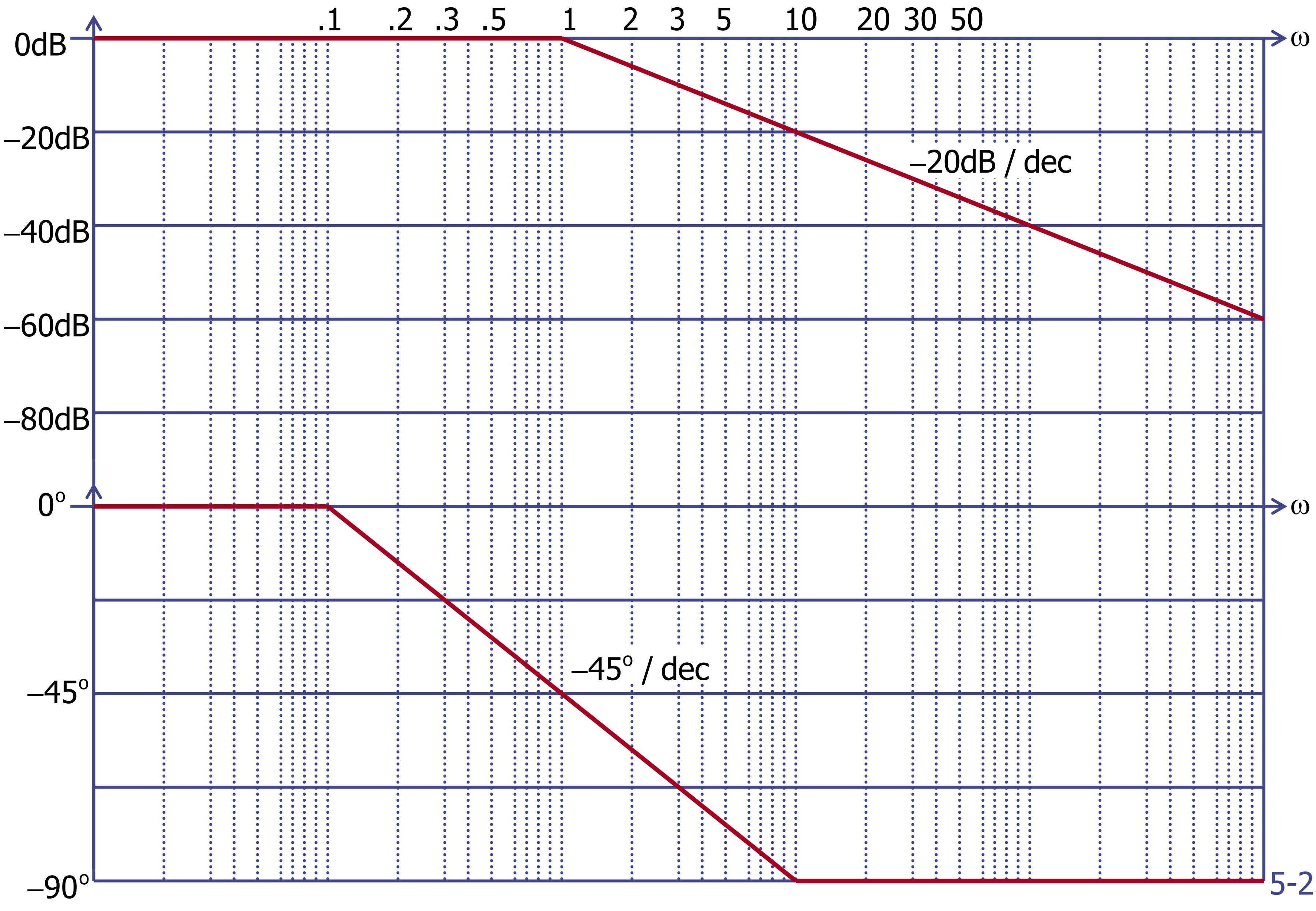
and

$$20\log|H(j\omega)| = -20\log|H'(j\omega)|$$

$$\angle H(j\omega) = -\angle H'(j\omega)$$



1^{st} Order Pole in Log-Linear Paper for p_1=1 rad/s



Standard Form of Transfer Function for Bode Plots

Recall a transfer function is written as

$$H(s) = H_0 \frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

For convenience in working out Bode plots, it is better to write the transfer function as

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = H_0' \frac{(1+s/z_1)(1+s/z_2)\ldots(1+s/z_m)}{(1+s/p_1)(1+s/p_2)\ldots(1+s/p_n)}$$

where H_0'= H(j0) is the DC gain. Note that H_0' is not the same as H_0.

The magic of Bode plots is that if we have a complicated transfer function as the one given above, we can break it down into smaller pieces, and add up all the pieces together one by one.

Example 5-2

Example 5-2: Sketch the Bode plots of

$$H(s) = \frac{10000(s+10)}{(s+1)(s+1000)}$$

Soln.:

(1) First of all, write H(s) in the standard form, i.e.,

$$H(s) = \frac{10000 \times 10(1+s/10)}{(1+s) \times 1000(1+s/1000)}$$

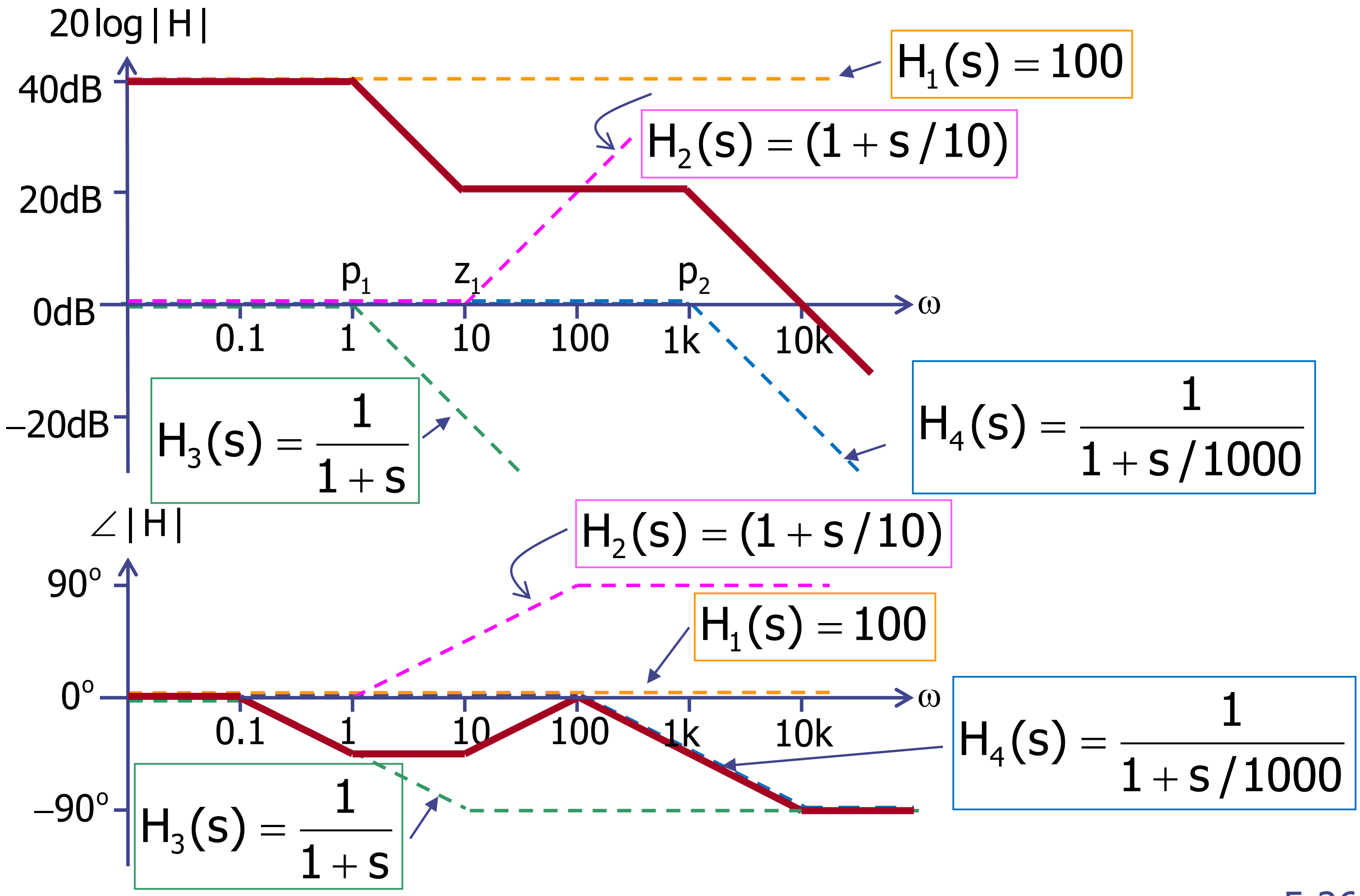
$$= \frac{100(1+s/10)}{(1+s)(1+s/1000)}$$

(2) Identify all corner frequencies: z_1=10 rad/s, p_1=1 rad/s, p_2=1000 rad/s.

(3) H(j0) = 100, and is equal to 40 dB.

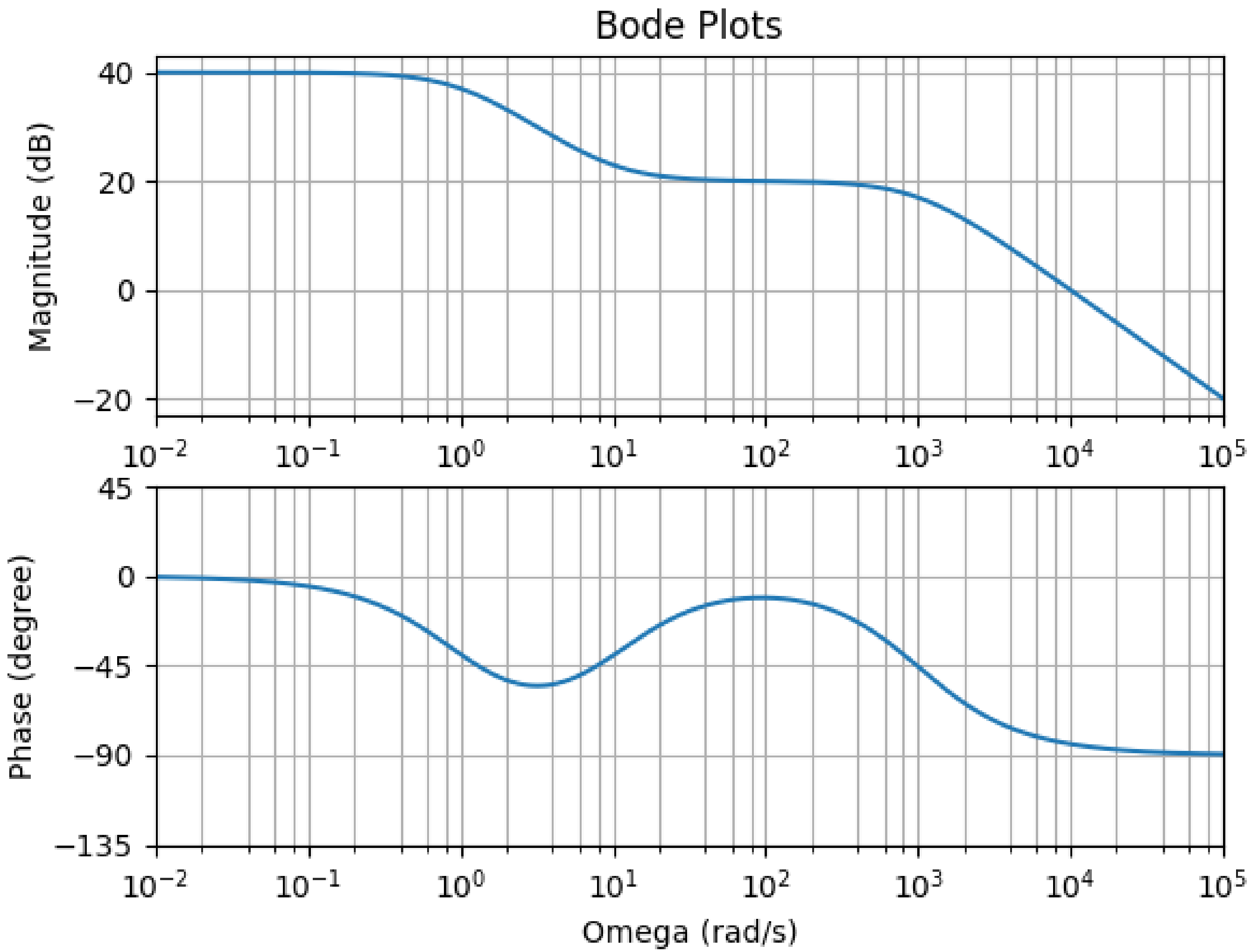
Example 5-2 (cont.)

(4) Draw Bode plots of individual factors, and start adding the Bode plots from low to high frequency.



Example 5-2 (cont.)

$$H(s) = \frac{10000(s+10)}{(s+1)(s+1000)}$$



Example 5-3

Example 5-3: Sketch the Bode plots of

$$H(s) = \frac{600}{(s+1)(s+30)}$$

Soln.:

(1) Write H(s) in the standard form:

$$H(s) = \frac{20}{(1+s)(1+s/30)}$$

Note that:

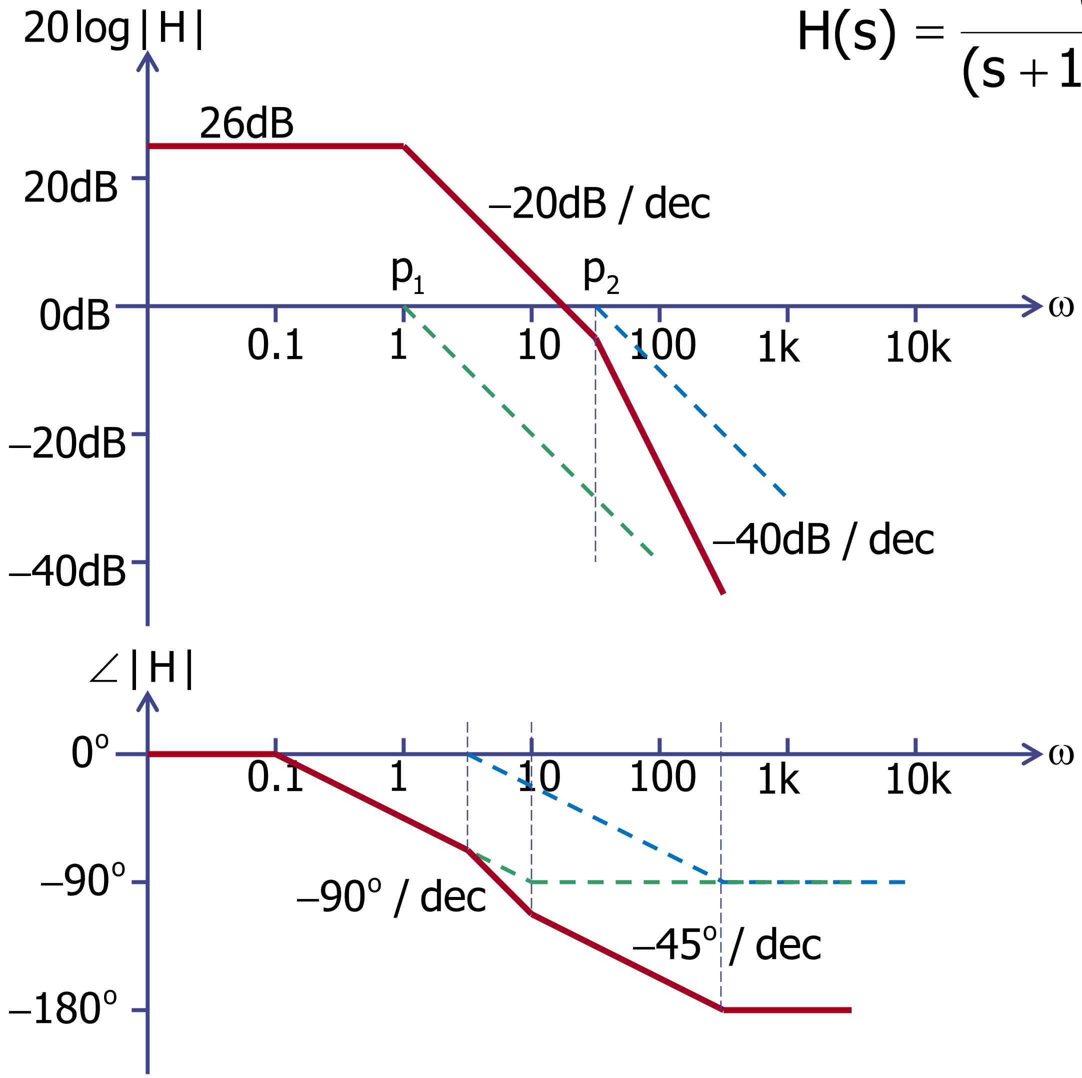
(1) $H(j0) = 20$, and is equal to 26 dB.

(2) For $\omega > 0$ rad/s, two –20 dB/dec lines add up to give one –40 dB/dec line. The phase plots are added similarly.

(3) At $\omega = 1$, the actual curve gives $|H(j1)| = 26 - 3 = 23$ dB

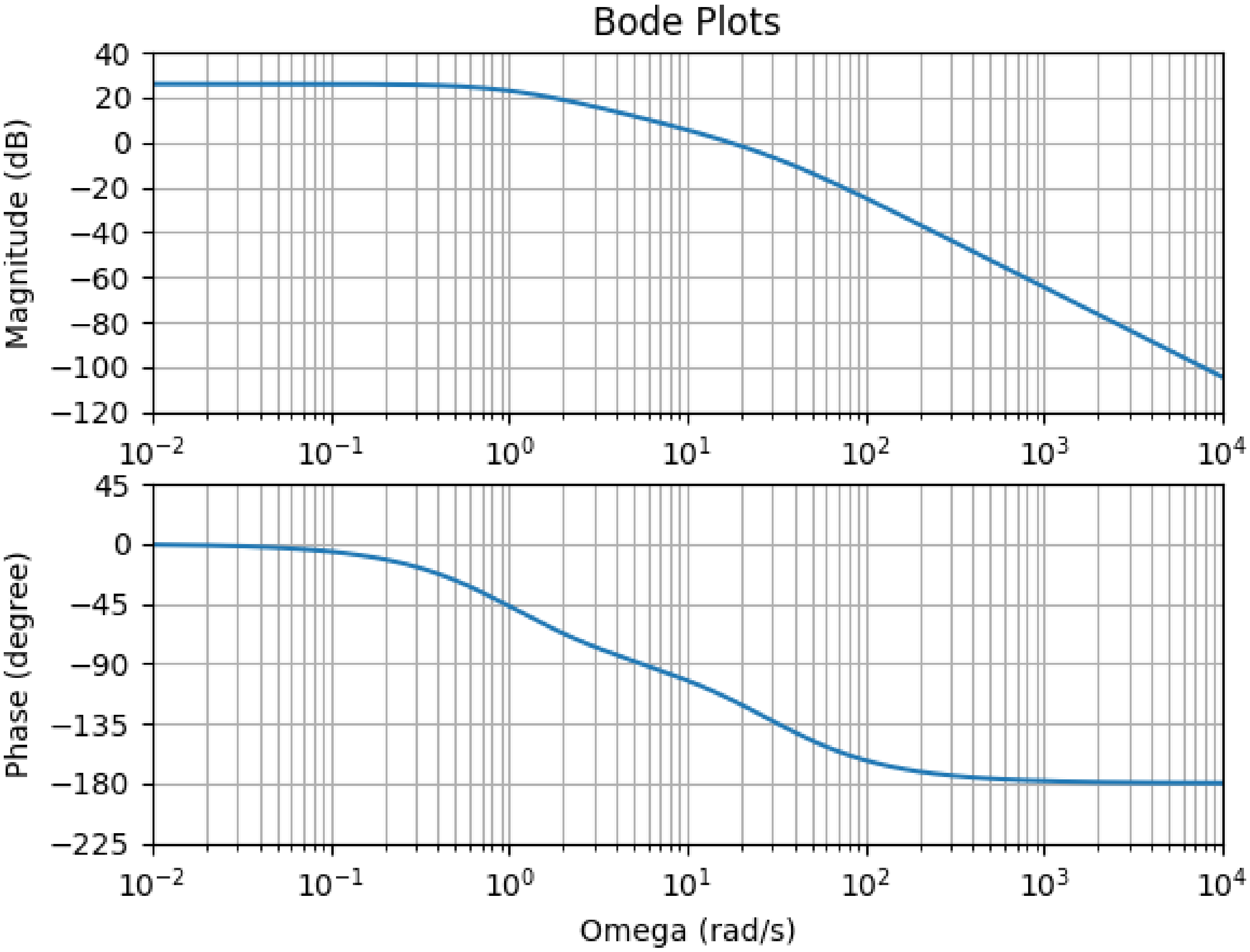
Example 5-3 (cont.)

$$H(s) = \frac{600}{(s+1)(s+30)}$$



Example 5-3 (cont.)

$$H(s) = \frac{600}{(s+1)(s+30)}$$



Bode Plots for a Pole or Zero at ω=0

For $$H_1(s) = \frac{s}{z_1}$$

Quick computation gives:

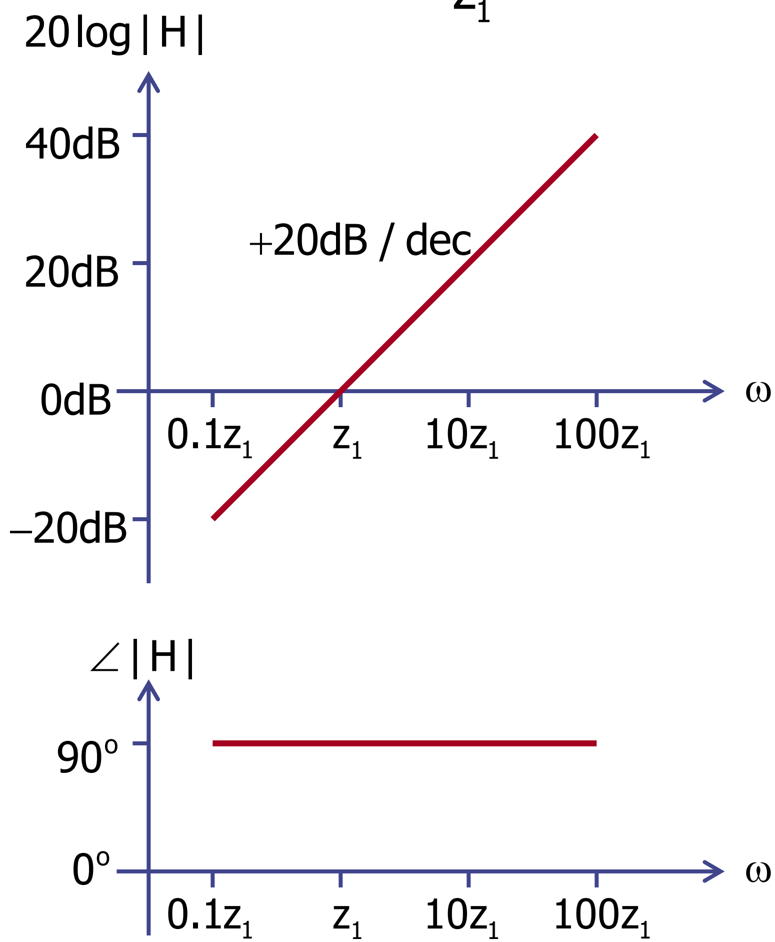
$H(j\omega)$	$\lvert H(j\omega) \rvert$	$20\log\lvert H(j\omega)\rvert$	$\angle H(j\omega)$
$H(j0.1z_1)$	0.1	-20dB	90^o
$H(jz_1)$	1	0dB	90^o
$H(j10z_1)$	10	$+20$dB	90^o

For $$H_2(s) = \frac{p_1}{s}$$

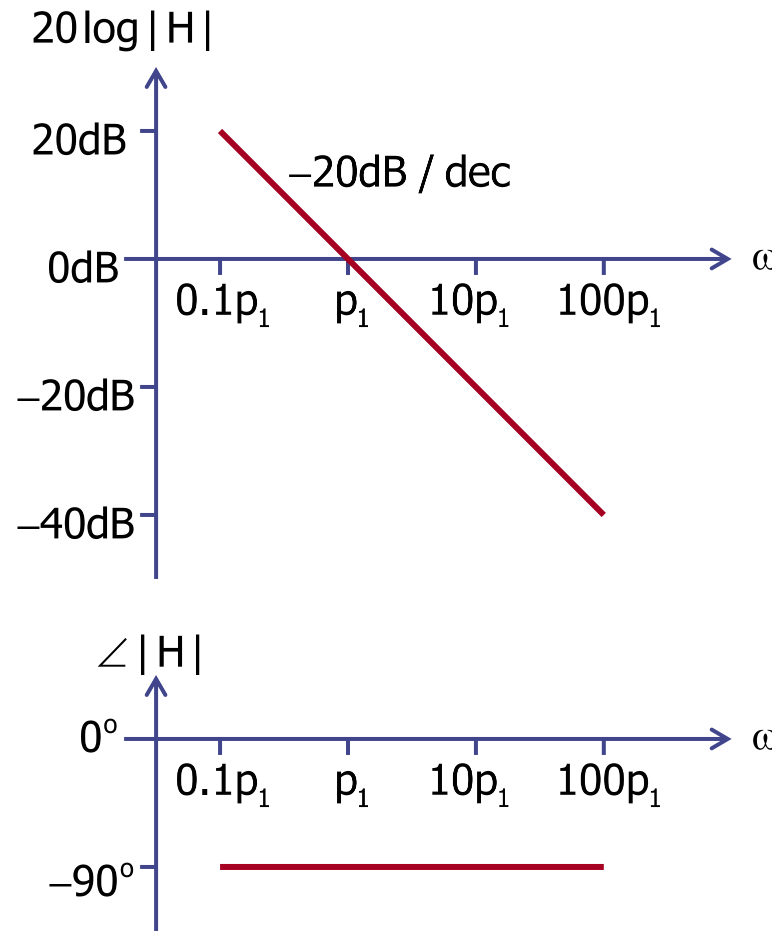
we have
$$\begin{aligned} 20\log\lvert H_2(j\omega)\rvert &= -20\log\lvert H_1(j\omega)\rvert \\ \angle H_2(j\omega) &= -\angle H_1(j\omega) \end{aligned}$$

Bode Plots of a Pole or Zero at ω=0 (cont.)

$$H_1(s) = \frac{s}{z_1}$$



$$H_2(s) = \frac{p_1}{s}$$



Example 5-4

Example 5-4: Sketch the Bode plots of

$$H(s) = \frac{60k \times s}{(s+200)(s+20k)}$$

Soln.:

(1) Write H(s) in the standard form:

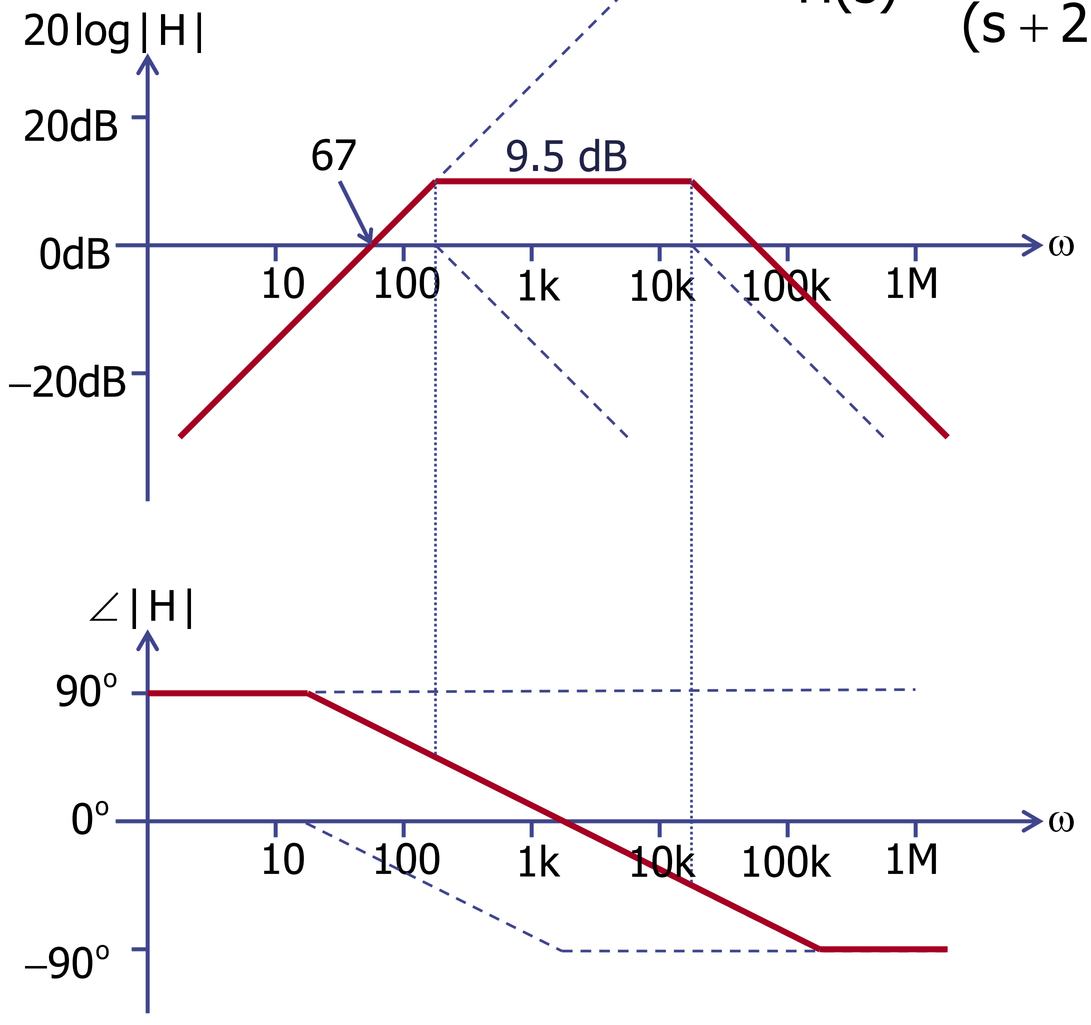
$$H(s) = \frac{s}{67}\frac{1}{(1+s/200)(1+s/20k)}$$

(2) This is a bandpass function (more discussions later) and one useful information is the mid-band frequency computed at ≈10X of the first corner frequency, i.e., s=j2k, (and the effect of the second corner frequency can be neglected):

$$|H(j2k)| \approx \left|\frac{j2k}{67}\frac{1}{(1+j2k/200)}\right| \approx \frac{200}{67} = 3 \rightarrow 9.5\text{ dB}$$

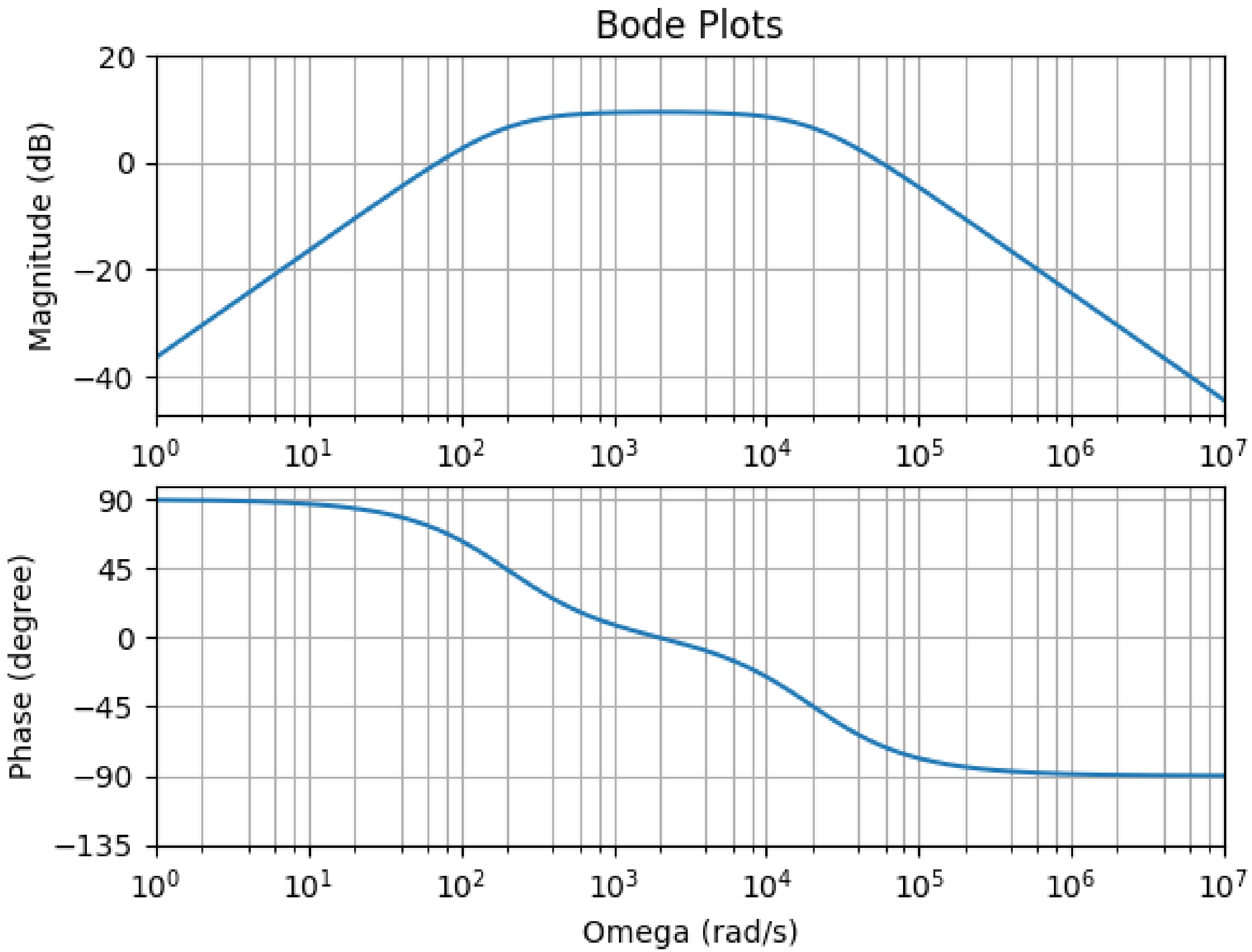
Example 5-4 (cont.)

$$H(s) = \frac{60k \times s}{(s+200)(s+20k)}$$



Example 5-4 (cont.)

$$H(s) = \frac{60k \times s}{(s+200)(s+20k)}$$



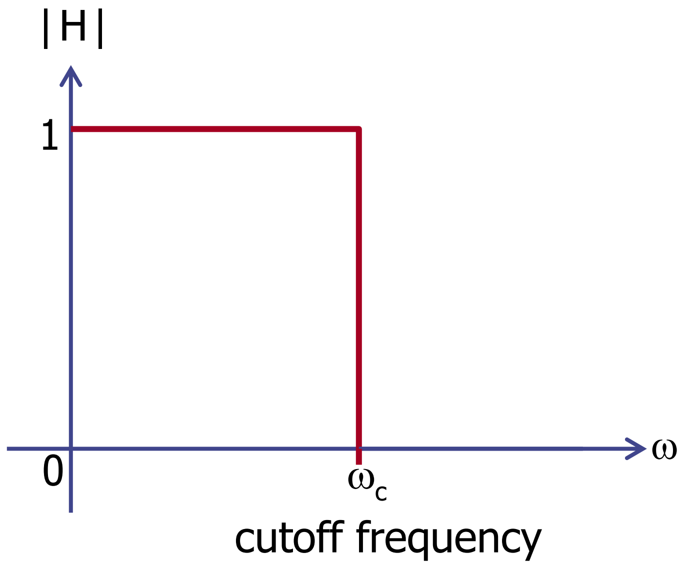
5.3.1 Filters

An electrical filter is a circuit that allows electrical signals of selected frequencies to pass through, while the rest will be attenuated (filtered out).

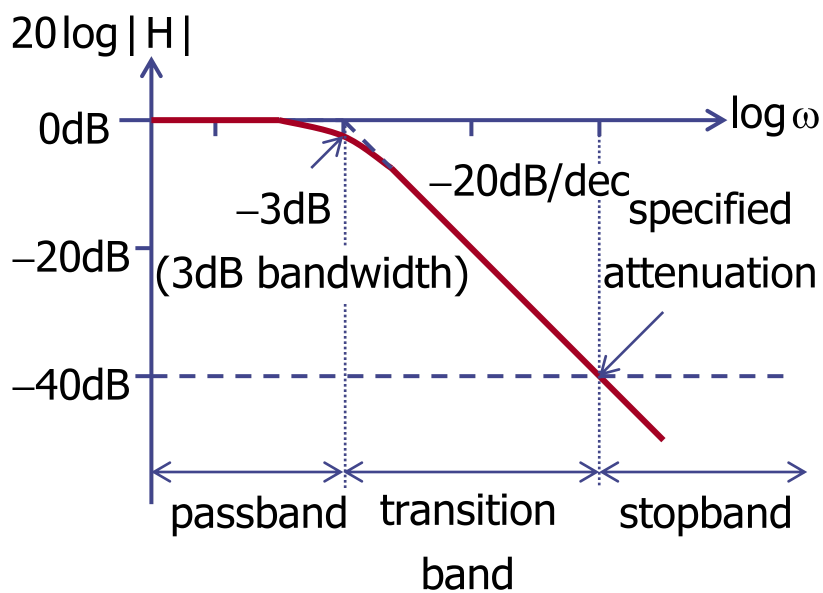
Some common types of filters are the lowpass filter, highpass filter, bandpass filter, bandstop (notch) filter, allpass filter, etc.

Lowpass Filter

Ideal Lowpass Filter



RC Lowpass Filter

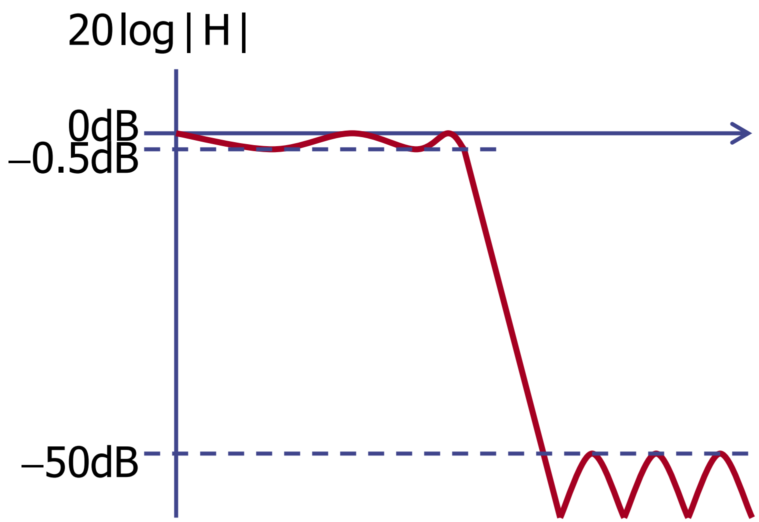


All frequencies below the cutoff frequency ($\omega<\omega_c$) have unity gain; all frequencies higher than ω_c are attenuated to zero.

The passband ripple of the RC lowpass filter may be specified as –3 dB, and the stopband attenuation may be specified as –40 dB.

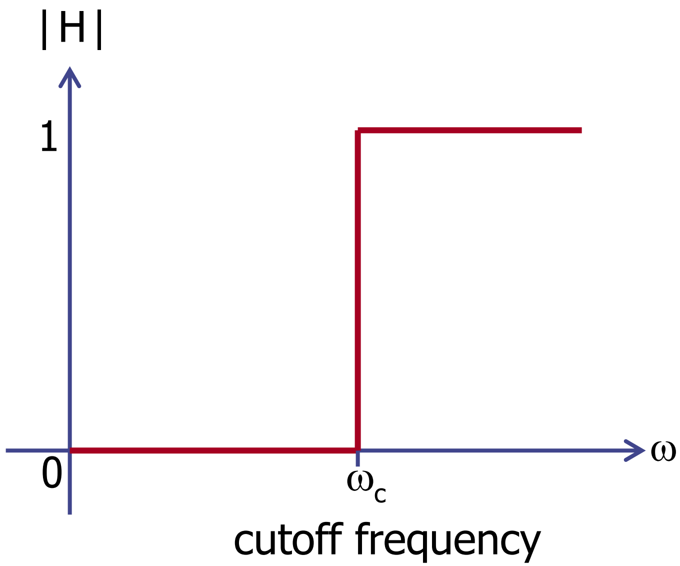
High Order Lowpass Filter

The RC filter is a first order lowpass filter that is far from being an ideal lowpass filter, which can only be approximated by high order transfer functions. A typical high order lowpass filter is shown below.

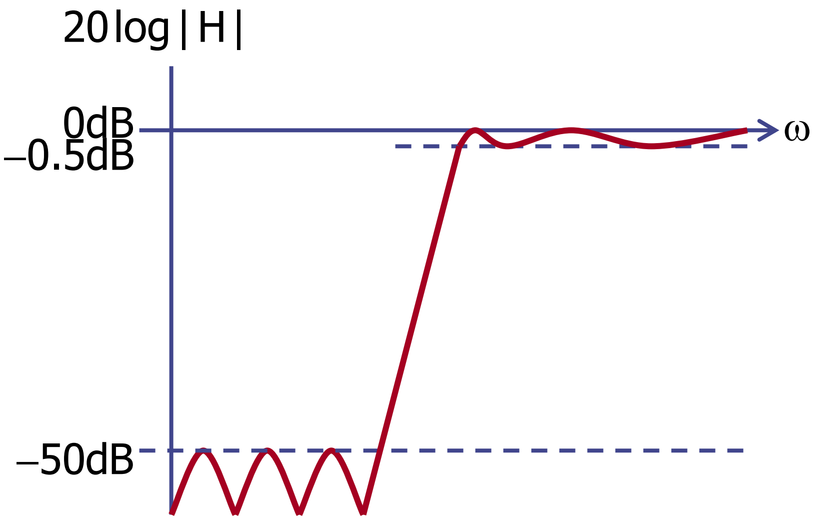


Highpass Filter

Ideal Highpass Filter

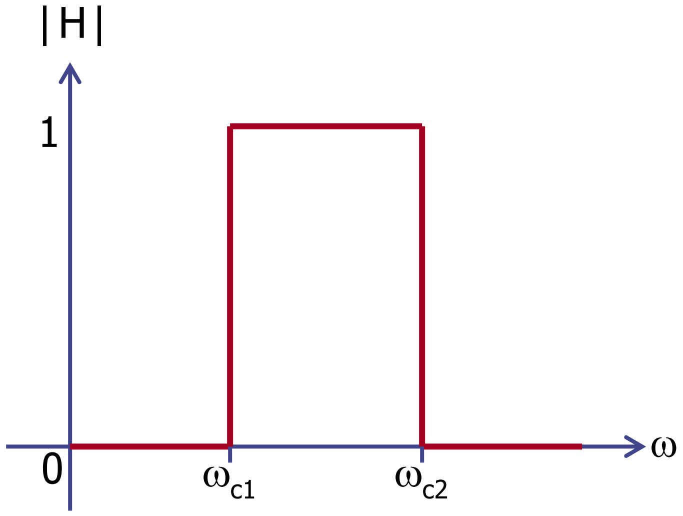


Practical Highpass Filter

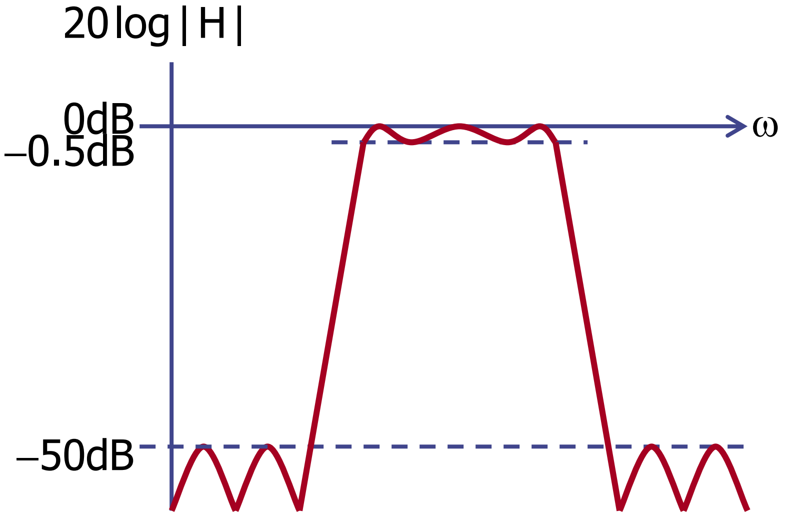


Bandpass Filter

Ideal Bandpass Filter

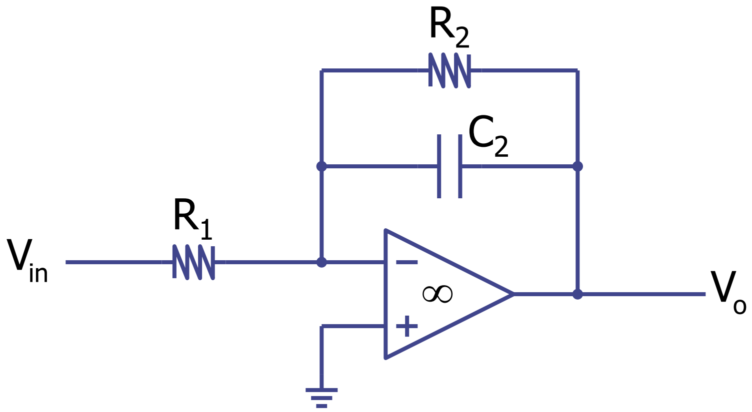


Practical Bandpass Filter

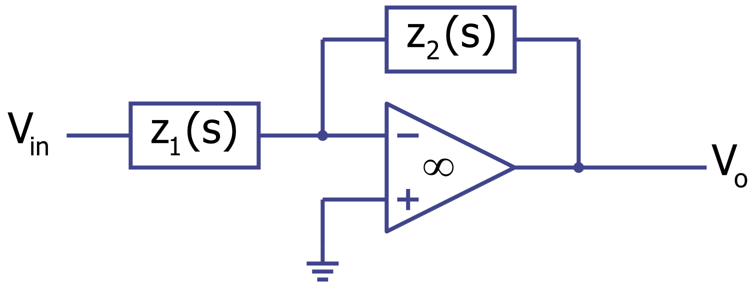


Lowpass Filter: Schematic

A first order lowpass filter can be constructed by adding a resistor across the capacitor of the integrator.



In fact, the analysis of this type of circuits can be generalized as follows:



$$H_1(s) = \frac{V_o(s)}{V_{in}(s)} = -\frac{z_2(s)}{z_1(s)}$$

where the Laplacian variable s is used in place of $j\omega$ ($s=j\omega$).

Lowpass Filter: Transfer Function

For the lowpass filter, the transfer function is

$$H_1(s) = \frac{V_o(s)}{V_{in}(s)} = -\frac{R_2 \,||\, (1/sC_2)}{R_1} = -\frac{1}{R_1}\frac{R_2 \times (1/sC_2)}{R_2 + (1/sC_2)}$$

$$= -\frac{R_2}{R_1}\frac{1}{1 + sC_2R_2}$$

We may compare this equation with that of Example 5-1. In this case, it is convenient to define $H(s) = -H_1(s)$, with $H_o = R_2/R_1$ as the DC gain and $p_1 = 1/C_2R_2$ as the pole. Hence,

$$H(j\omega) = H_o \frac{1}{1 + \frac{j\omega}{p_1}}$$

Lowpass Filter: Frequency Response

Compute the transfer function $H(j\omega)$ at various frequencies (frequency response):

$$H(jp_1/100) = \frac{H_o}{1+j0.01} = H_o \times 1.000\angle -0.57^o \qquad 0dB$$

$$H(jp_1/10) = \frac{H_o}{1+j0.1} = H_o \times 0.995\angle -5.71^o \qquad 0dB$$

$$H(jp_1/3) = \frac{H_o}{1+j/3} = H_o \times 1.054\angle -18.4^o \qquad 0dB$$

$$H(jp_1) = \frac{H_o}{1+j} = H_o \times 0.707\angle -45.0^o \qquad -3dB$$

$$H(j3p_1) = \frac{H_o}{1+j3} = \frac{H_o}{3.162}\angle -71.6^o \qquad -10dB$$

$$H(j10p_1) = \frac{H_o}{1+j10} = \frac{H_o}{10.05}\angle -84.29^o \qquad -20dB$$

$$H(j100p_1) = \frac{H_o}{1+j100} = \frac{H_o}{100.0}\angle -89.43^o \qquad -40dB$$

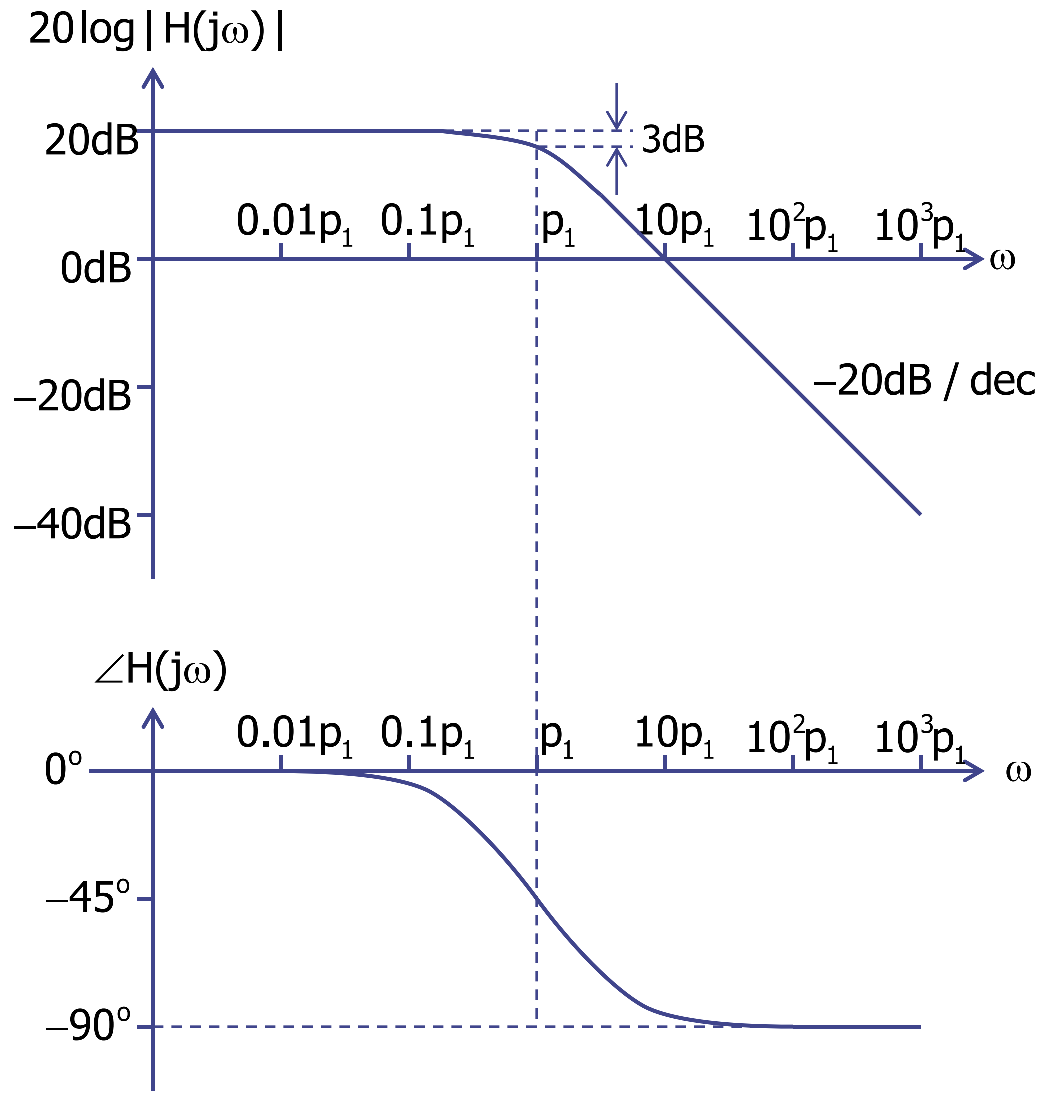
Lowpass Filter: Bode Plots

$$H(s) = H_o \frac{1}{1 + \frac{s}{p_1}}$$

Magnitude plot:

(Assume H_o = 10, or 20 dB)

Phase plot:

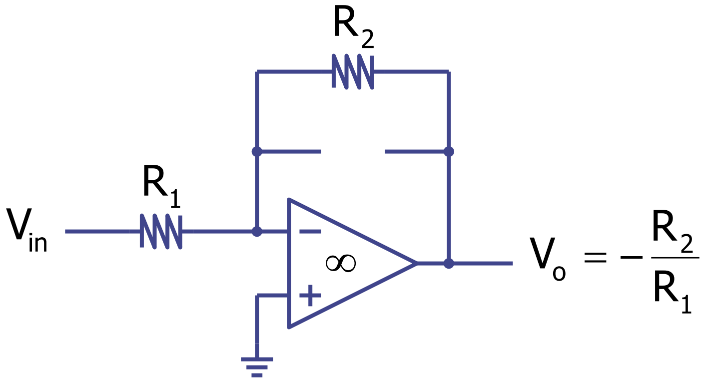


Lowpass Filter: Qualitative Analysis

The lowpass filter allows low-frequency signals to pass to the output V_o without much attenuation, while high-frequency signals are blocked, that is, not allowed to pass through.

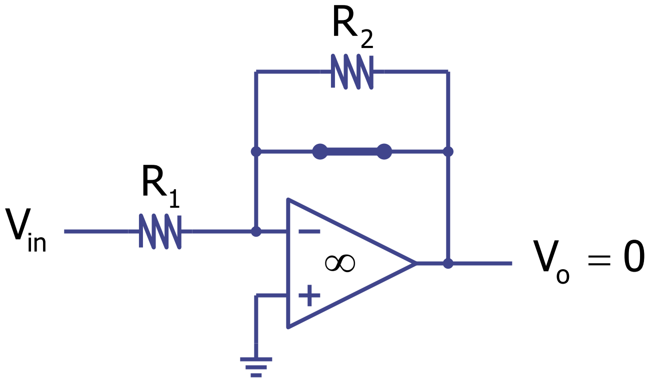
One simple way to identify the type of filters is to consider the circuit at very low frequency such that capacitors are considered as open circuits; and at very high frequency such that capacitors are considered as short circuits.

Low frequency signals with gain $-R_2/R_1$:



Inverting amplifier

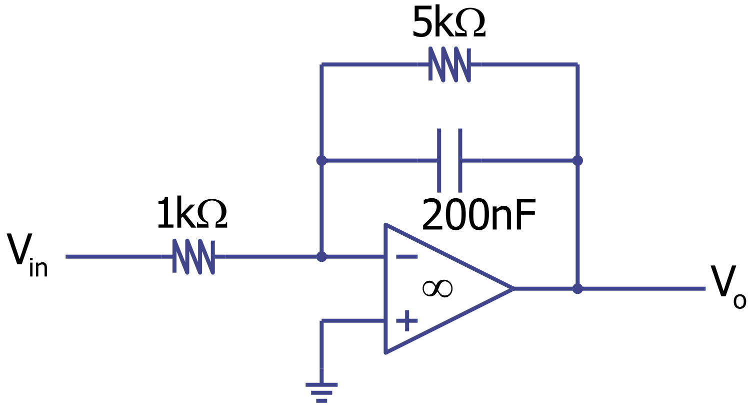
High frequency signals attenuated to zero:



Inverting amplifier with zero gain

Example 5-5

Example 5-5: Sketch the output waveform w.r.t. $V_{in}(t)$ for (1) $V_{in}(t)$ = 2sin(1kt) V; and (2) $V_{in}(t)$ = 2sin(10kt) V.



Soln.:

The transfer function is

$$H(s) = \frac{V_o(s)}{V_{in}(s)} = -\frac{R_2}{R_1}\frac{1}{1+sC_1R_2} = -\frac{5k}{1k}\frac{1}{1+s\times 200n\times 5k}$$

$$= -\frac{5}{1+\dfrac{s}{1k}}$$

Example 5-5 (cont.)

(1)
$$H(1\text{k}j) = -\frac{5}{1+\frac{1\text{k}j}{1\text{k}}} = -\frac{5}{1+j} = -\frac{5}{\sqrt{2}}\angle(-45°) = \frac{5}{\sqrt{2}}\angle 135°$$

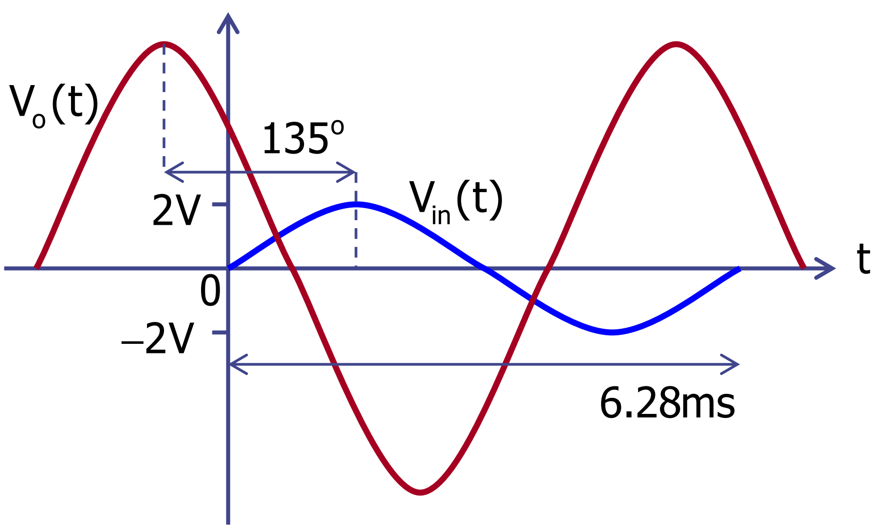
Phase Shift

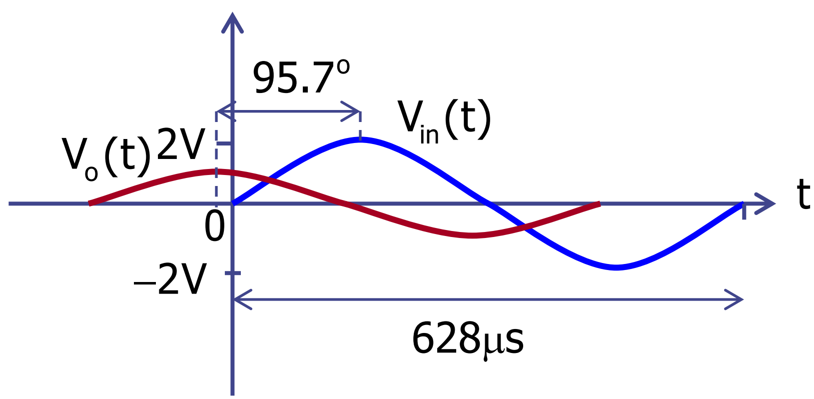
Gain

$$V_o(t) = 2\frac{5}{\sqrt{2}}\sin(1\text{k}t + 135°) = 7.07\sin(1\text{k}t + 135°)$$

(2)
$$H(10\text{k}j) = -\frac{5}{1+\frac{10\text{k}j}{1\text{k}}} = -\frac{5}{1+10j} = -\frac{5}{\sqrt{101}}\angle(-84.3°) = \frac{5}{\sqrt{101}}\angle 95.7°$$

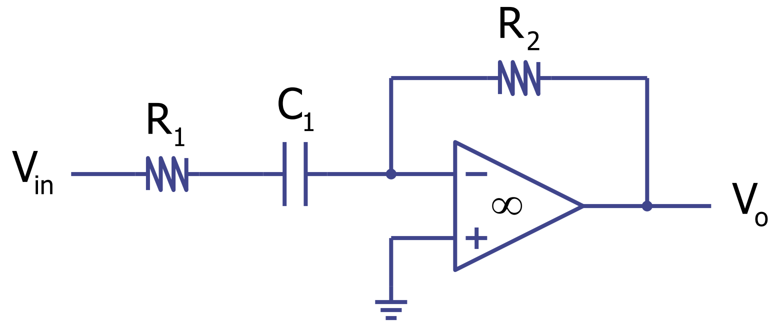
$$V_o(t) = 2\frac{5}{\sqrt{101}}\sin(10\text{k}t + 95.7°) = 0.995\sin(10\text{k}t + 95.7°)$$





Highpass Filter

Schematic of highpass filter:



Transfer function of highpass filter:

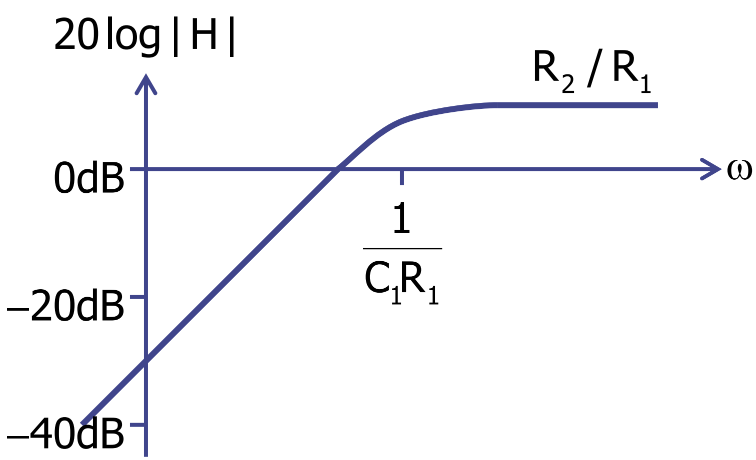
$$H(s) = \frac{V_o(s)}{V_{in}(s)} = -\frac{R_2}{R_1 + 1/sC_1} = -\frac{sC_1R_2}{1 + sC_1R_1}$$

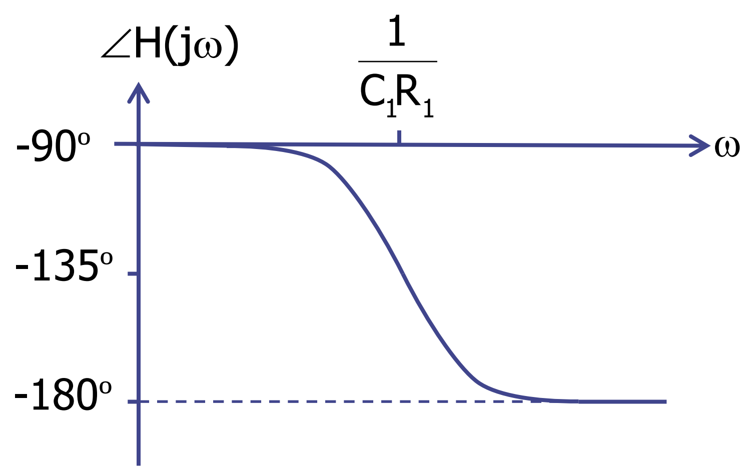
At low frequencies,

$$H(j\omega) \approx -j\omega C_1 R_2 \approx 0$$

At high frequencies,

$$H(j\omega) \approx -\frac{R_2}{R_1}$$



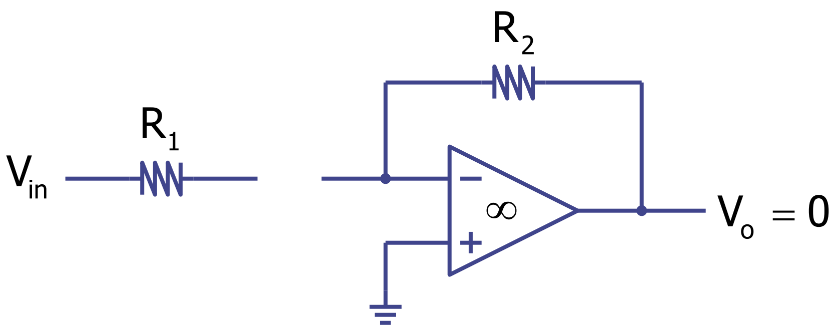


Highpass Filter: Qualitative Analysis

The highpass filter blocks low-frequency signals but allows high-frequency signals to pass to the output V_o without much attenuation.

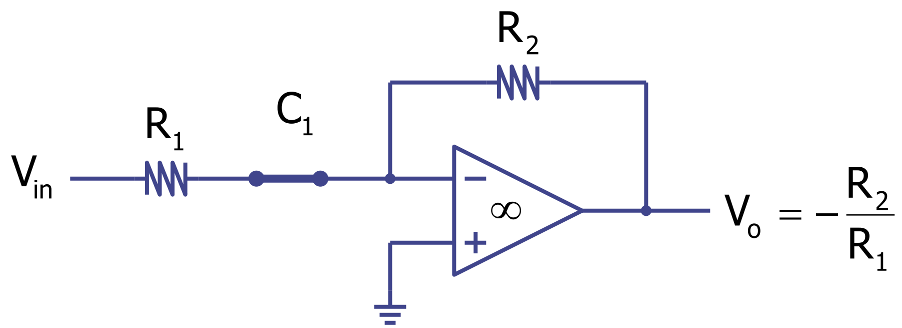
One simple way to identify the type of filters is to consider the circuit at very low frequency such that capacitors are considered as open circuits; and at very high frequency such that capacitors are considered as short circuits.

Low frequency signals attenuated to zero:



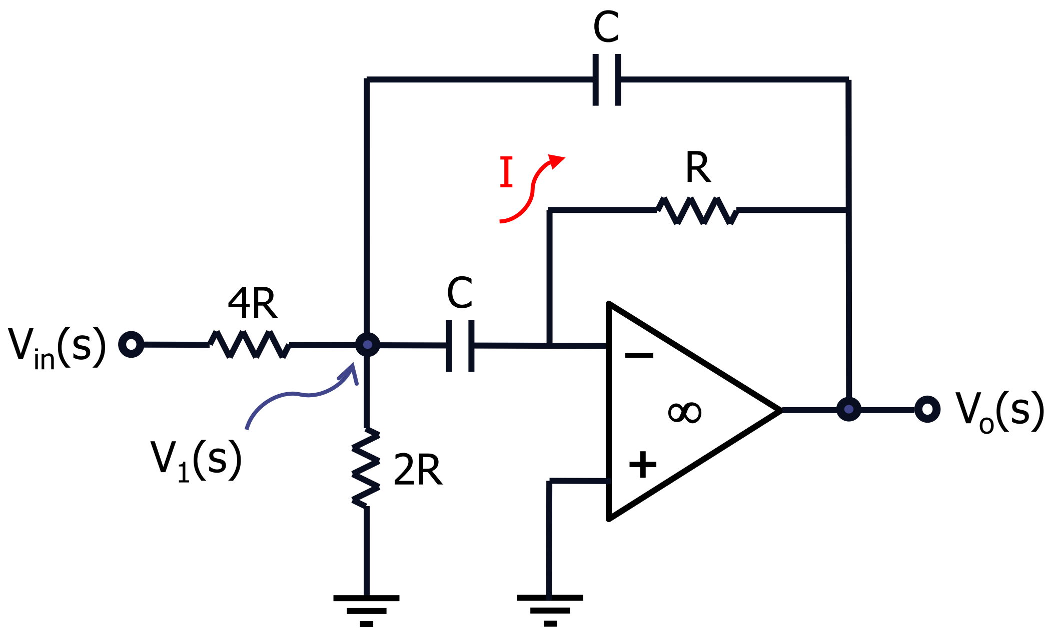
Unity gain buffer with zero input

High frequency signals with gain $-R_2/R_1$:



Inverting amplifier

Bandpass Filter



V_ is a virtual ground. The same current I goes through C and R. Hence

$$I = \frac{V_1}{\frac{1}{sC}} = -\frac{V_o}{R} \qquad V_1 = -\frac{V_o}{sRC}$$

Apply KCL to node V_1:

$$\frac{V_1 - V_{in}}{4R} + \frac{V_1}{2R} + \frac{V_1}{\frac{1}{sC}} + \frac{V_1 - V_o}{\frac{1}{sC}} = 0$$

Bandpass Filter (contd.)

From before

$$\frac{V_1 - V_{in}}{4R} + \frac{V_1}{2R} + \frac{V_1}{\frac{1}{sC}} + \frac{V_1 - V_o}{\frac{1}{sC}} = 0$$

$$V_1 - V_{in} + 2V_1 + 4sRCV_1 + 4sRC(V_1 - V_o) = 0$$

$$V_1(3 + 8sRC) - 4sRCV_o = V_{in}$$

Substituting previous result

$$V_1 = -\frac{V_o}{sRC}$$

We have

$$-\frac{V_o(3 + 8sRC)}{sRC} - 4sRCV_o = V_{in}$$

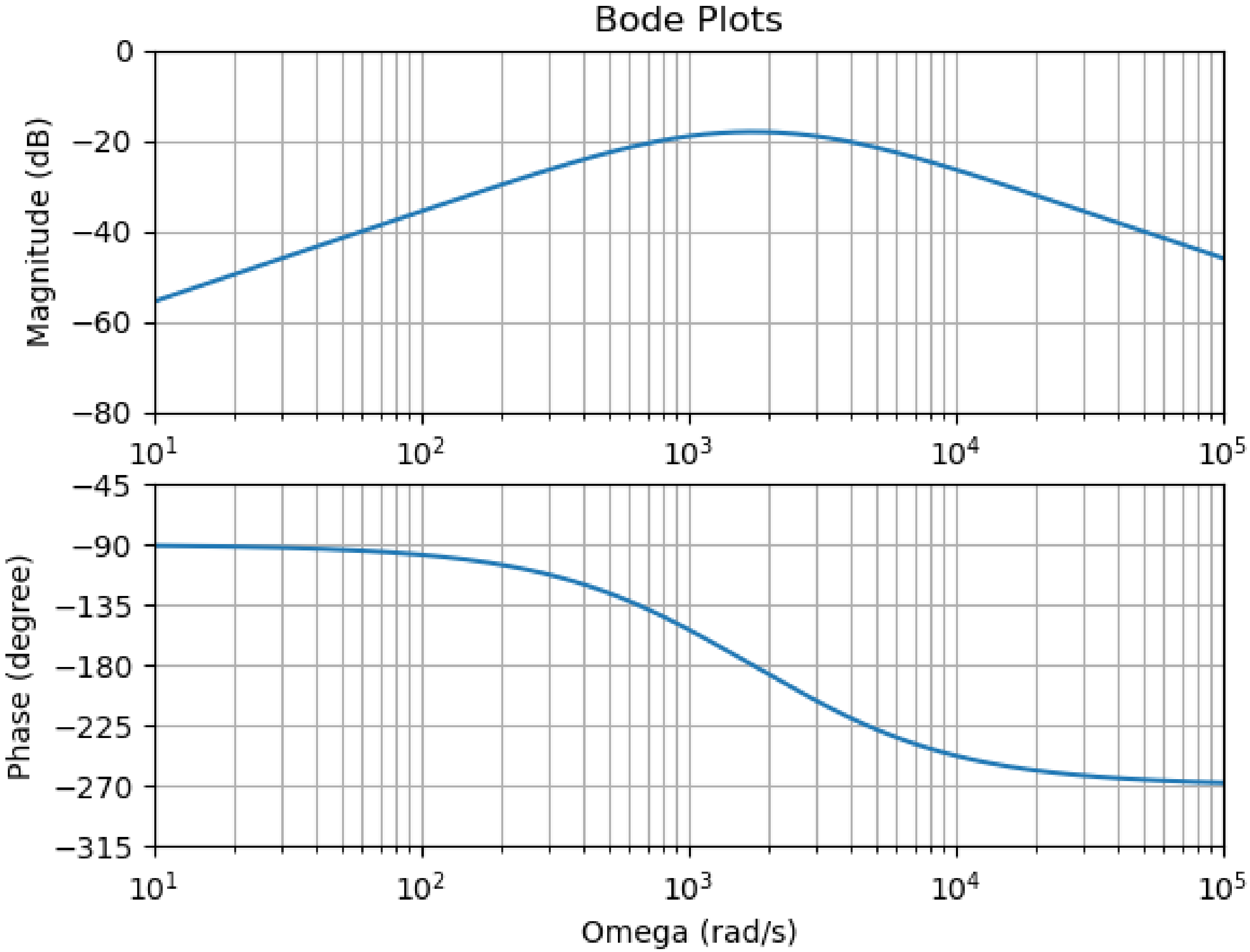
Finally,

$$H(s) = \frac{V_o(s)}{V_{in}(s)} = -\frac{sRC}{3 + 8sRC + 4s^2R^2C^2} = -\frac{sRC}{(3 + 2sRC)(1 + 2sRC)}$$

Bandpass Filter (contd.)

Design example:
R = 500 Ω, C = 1 mF
RC = 0.5 ms

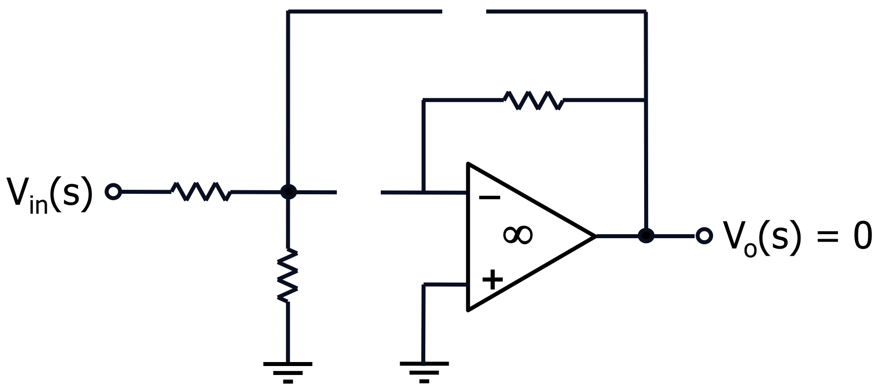
$$H(s) = -\frac{500s}{(s+3000)(s+1000)}$$



Bandpass Filter: Qualitative Analysis

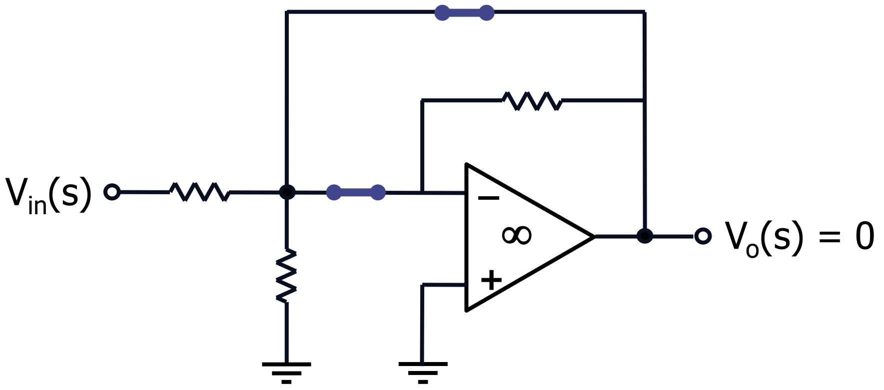
The bandpass filter blocks both low- and high-frequency signals but allows mid-frequency signals to pass to the output V_o.

Low frequency signals attenuated to zero:



Unity gain buffer with zero input

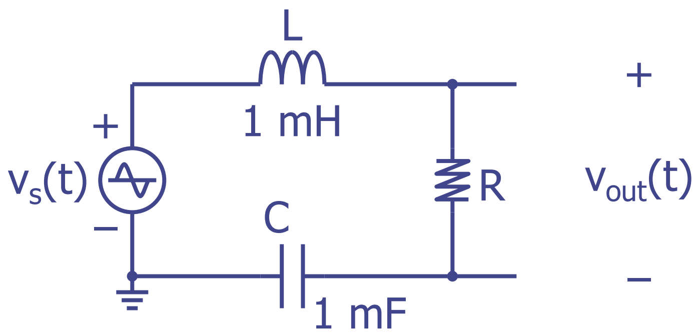
High frequency signals attenuated to zero:



Inverting amplifier with zero gain

5.3.2 Second Order Resonance

Example 5-6: Consider an RLC circuit with its input connected to a sinusoidal signal generator. Find the transfer function H(s) = $V_{out}(s)/V_{in}(s)$ and explore the Bode plots for various values of R.



Soln.: $$V_{out}(s) = \frac{R}{sL + R + \frac{1}{sC}} V_{in}(s) = \frac{sRC}{s^2LC + sRC + 1} V_{in}(s)$$

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{sRC}{s^2LC + sRC + 1}$$

Second Order System

The loop current is limited by the capacitor impedance $-j/(\omega C)$ at low frequency and the inductor impedance $j\omega L$ at high frequency. The loop current, and hence V_{out}, reaches a maximum when these impedances cancel each other. This occurs at $\omega = \frac{1}{\sqrt{LC}}$.

Example 5-6 (contd.)

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{sRC}{s^2LC + sRC + 1}$$

Second Order System

The transfer function has a zero at s = 0. There are two poles obtainable by solving

$$s^2LC + sRC + 1 = 0$$

$$s = \frac{-RC \pm \sqrt{(RC)^2 - 4LC}}{2LC} = \frac{-R \pm \sqrt{R^2 - 4\frac{L}{C}}}{2L}$$

Case 1: $R > 2\sqrt{\frac{L}{C}}$ Two unequal real poles

Case 2: $R = 2\sqrt{\frac{L}{C}}$ Two equal real poles

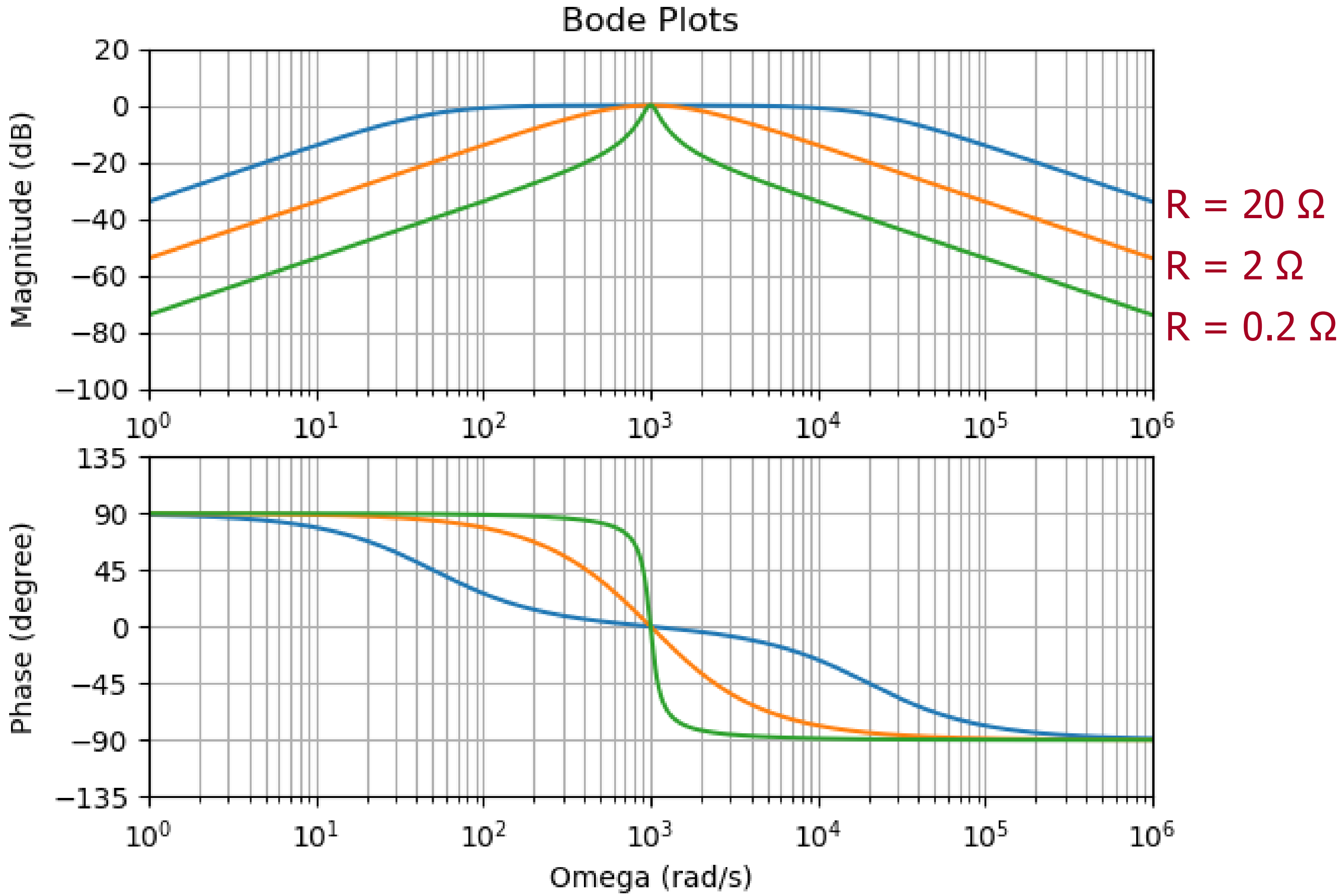
Case 3: $R < 2\sqrt{\frac{L}{C}}$ Two complex conjugate poles

Example 5-6 (cont.)

$$H(s) = \frac{sRC}{s^2LC + sRC + 1}$$

L = 1 mH, C = 1 mF

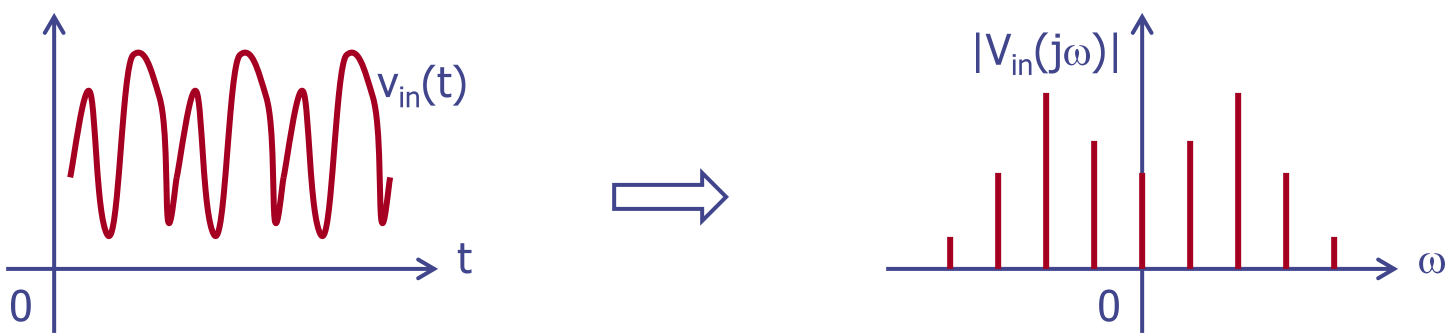
Resonant frequency = $\frac{1}{\sqrt{LC}}$ = 1000 rad/s



As R decreases, the two real poles of H(s) merge to become two complex conjugate poles. A phenomenon known as resonance occurs.

Frequency Response for Arbitrary Signals

In general, a signal which is energy limited can be decomposed into its frequency components. This can be done via the Fourier Transform. Each frequency component can be thought as a sinusoidal wave with specific amplitude and phase. Mathematically, we consider both positive and negative frequencies (recall that $\cos(\omega t) = \frac{1}{2}(e^{+j\omega t} + e^{-j\omega t})$).



Afterward we can perform the frequency response analysis to all the frequency components in a signal.